Chapter 1

Sets and Counting

1.1 Notation

1.1.1 Definition

A set is a collection of elements, or objects.

1.1.2 Set Notation

A set is always abbreviated with a capital letter (Sets are usually denoted by upper case letters $A, B, C, D, \ldots$). The members or elements of a set are requested by lower case letters'; $a, b, c, d, \ldots$ and are enclosed in braces, $\{\}$.

A member of a set is denoted by the symbol $\in$. Hence $x \in A$ means x is a member of the set A. The opposite or negative of any symbol is obtained by drawing the symbol / (slash) through the given symbol. Thus $x \notin A$ is read "x is not a member of the set A."

1.2 Basic Concepts and Symbolism

1.2.1 Symbolism

There are two ways of representing sets:

(a) Tabular notation/listing members

(b) Set-builder notation

(1) Tabular Notation
A set can be represented by **listing its members**, such a set is said to be defined. This is called the tabular notation presented as $x, y, \ldots$ and read "the set whose members are $\{x, y \ldots\}$".

Example 1.2.1 Write the following sets in tabular notation:

(a) $A = \{x : x^2 + 3x + 2 = 0\}$

(b) $W = \{\text{even numbers less than } 10\}$

(c) $A = \{\text{colors of a Ugandan flag}\}$

(d) $B = \{x : x \text{ is a letter in the word Uganda}\}$

(e) $C = \{x : x \text{ is a consonant}\}$

1.4.5 Disjoint Sets

Two sets a and B are said to be disjoint if they have any members in common. That is $A \cap B = \phi$.

Example 1.4.7 (a) Let $A = \{4, 5, 6\}$ and $B = \{1, 3, 7\}$ then A and B are disjoint since $A \cap B = \phi$.

Example 1.4.8 Let $A = \{x : -1 \leq x \leq 3\}$ and $B = \{5, 10, 15, \ldots\}$. The two sets have no members in common, the sets are therefore disjoint.

1.4.6 Cardinality

The cardinality of a set is the number of elements in that set. For example

(i) If $A = \{a, c, d\}, \implies |A| = 3$

(ii) If $B = \{1, 0, 1, -1\}, \implies |B| = 3$

(iii) If $C = \{9, 12, -3, 0\}, \implies |C| = 4$

(iv) If $A = \{\}, \implies |A| = 0$

1.5 Set Operations

Two sets A and B are said to be comparable if either a is a subset of $B(A \subseteq B)$ or B is a subset of $A(B \subseteq A)$ that is, if one of the sets is a subset of the other. Otherwise sets a and B are non-comparable. If the sets A is non-comparable to set B, the there is at least one element of the set A that is not in the set B and there is also an element in the set B that is not in the set A.

Example 1.5.1 (a) Let $A = \{a, b, c\}$ and $B = \{q, b, c, d\}$. Set A is comparable to set b since set A is a subset of set B, that is, $A \subset B$.

(b) Let $R = \{1, 2\}$ and $S = \{2, 3, 4\}$. Sets R and S are non comparable since $1 \in R$ but $1 \notin S$ and also $3 \in S$ but $3 \notin R$.

1.5.1 Intersection of Sets

Given any two sets A and B. The collection of all members that belong to both A and B is called the intersection of A and B. The intersection of A and B is denoted by $A \cap B$, and read "$A \cap B$" or $A \cap B$". In the set-builder notation the intersection of A and B is written:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Example 1.5.2 $A = \{1, 2, 3, 4, 5\}, B = \{3, 4, 6\}$ Then $A \cap B = \{3, 4\}$.

Example 1.5.3 (a) If $A = \{p, q, r, s\}$ and $B = \{q, s, t\}$ then $A \cap B = \{q, s\}$.

(b) Let $A = \{a, b, c, d, e\}$ and $B = \{a, c, e, g\}$. Find $A \cap B$.
The elements that are in both sets are a, c, e, thus; $A \cap B = \{a, c, e\}$.

(c) Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$. Find $A \cap B$.
There are not elements that are in both sets. Therefore, $A \cap B = \phi$. The sets are disjoint.

down numbers over 3,000 years ago. Later, the Romans developed a system of numerals that used letters from their alphabet rather than special symbols. Today, we use numbers based on the Hindu-Arabic system. We can write down any number using combinations of up to 10 different symbols $(0, 1, 2, 3, 4, 5, 6, 7, 8,$ and $9)$. The ancient Egyptians developed number systems to keep accounts of what was bought and sold. There are many different ways we can classify numbers. In fact, many of the classifications are subsets of other classifications. A few especially common and important sets will be given their own names, i.e , their own symbols. But as we start to explore them, relax, numbers are fun !, numbers are your friend.

1.7.1 Natural numbers $\mathbb{N}$

The natural numbers are the positive integers ; hence

$$\mathbb{N} = \{1,\ 2,\ 3,\ \cdots\}$$

The natural numbers were the first numbers system developed and were used primarily, at one time, for counting.
Clearly, you can notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \Re$. Among the Natural numbers are the even and odd numbers;
Even numbers could be defined as the numbers divisible by 2
Or even numbers are those that can be divided evenly into groups of two. The number four can be divided into two groups of two.
Odd numbers can NOT be divided evenly into groups of two. The number five can be divided into two groups of two and one group of one. Even numbers always end with a digit of 0, 2, 4, 6 or 8. Numbers such as 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30 are even numbers. Alternatively Odd numbers always end with a digit of 1, 3, 5, 7, or 9. They include 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31. The natural numbers are only closed under the operation of addition and multiplication. The difference and quotient of two natural numbers need not to be a natural number ; e.g $2 - 7 = -5$, and $\frac{2}{7}$ are both not natural numbers. The Prime numbers are those natural numbers p ,excluding 1 , which are only divisible by 1 and p itself. We list the first few prime numbers: $2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19, \cdots$

1.7.2 Integers $\mathbb{Z}$

The Integers are those real numbers

$$\cdots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \cdots$$

We denote the integers by $\mathbb{Z}$; hence $\mathbb{Z} = \{\cdots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \cdots\}$ The Integers are also refered to as "whole" numbers. One important property of the Integers is that they are "closed" under the operations of addition, subtraction and multiplication, that is, the sum, difference and product of two integers in an integer. Notice that the quotient (division) of the integers, say 3 and 7, need not to be an integer, hence integers are not closed under division.

1.7.3 Rational numbers $\mathbb{Q}$

The *rational numbers* are those *real numbers* which can be expressed as the ratio of two *integers*. Accordingly,

$$\mathbb{Q} = \{x\ ;\ x = \frac{p}{q}\ ,\ \gcd(p, q) = 1,\ \text{where } p,\ q\ \in\ \mathbb{Z}\}$$

if and only if either yesterday was cloudy then will today be rainy, or today will be clear then either today is rainy but not cold or yesterday was cloudy

Example 2.1.16 Let the atomic formulas W, R and S be given by the following:

$$\begin{aligned} W &: \text{ It is windy} \\ R &: \text{ It is rainy} \\ S &: \text{ It is sunny} \end{aligned}$$

Translate the following English statements into $\mathcal{L}_P$ formulas (statements).

(a) If it is raining, then it is windy and the sun is not shining.

$$R \to (W \wedge \neg S)$$

(b) It is not raining, if either the sun is shining or it is not windy.

$$(S \vee \neg W) \to \neg R$$

(c) Either it's windy only if it is raining or its not sunny only if it's raining.

$$(W \to R) \vee (\neg S \to R)$$

(d) It is windy and not sunny only if it is raining.

$$(W \wedge \neg S) \to R$$

(e) Rain is a necessary condition for wind with no sunshine.

$$R \to (W \wedge \neg S)$$

It is only the conjunction statement (c) which is true since both the simple propositions are true. Each of the other composite propositions is false since at least one of the simple propositions is false.

2.2.5 Truth table for disjunction ($\vee$)

	α	β	$\alpha \vee \beta$
	T	T	T
	T	F	T
	F	T	T
*	F	F	F

Consider the following composite propositions and interpret the disjunction statements.

(a) Kampala is in Uganda or $3 + 3 = 7$

(b) Kampala is in America or $3 + 3 = 6$

(c) Kampala is in Uganda or $3 + 3 = 6$

(d) Kampala is in America or $3 + 3 = 7$.

It is only the disjunction statement a, b and c that are true since at least one of the simple propositions is true. The disjunction statement (d) is false because both the simple propositions are false.

2.2.6 Truth table for Bi-implication, iff ($\leftrightarrow$)

	α	β	$\alpha \leftrightarrow \beta$
*	T	T	T
	T	F	F
	F	T	F
*	F	F	T

Consider the following composite propositions and interpret the bi-conditional statements.

(a) Kampala is in Uganda if and only if $3 + 3 = 7$

(b) Kampala is in America if and only if $3 + 3 = 6$

(c) Kampala is in Uganda is and only if $3 + 3 = 6$

(d) Kampala is in America if and only if $3 + 3 = 7$

The bi-conditional propositions (c) and (d) are true because in (c) both simple propositions are true and in (d) both simple propositions a and b are false because the two simple propositions have opposite truth-values.

Example 2.2.2 Construct the truth table for $P \rightarrow (Q \rightarrow R)$

P	Q	R	$(Q \rightarrow R)$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Proposition 2.3 *If α and β are formulas and v is a truth assignment, then:*

1. $v(\neg\alpha) = T$ *if and only if* $v(\alpha) = F$.
2. $v(\alpha \to \beta) = T$ *if and only if* $v(\beta) = T$ *whenever* $v(\alpha) = T$;
3. $v(\alpha \wedge \beta) = T$ *if and only if* $v(\alpha) = T$ *and* $v(\beta) = T$;
4. $v(\alpha \vee \beta) = T$ *if and only if* $v(\alpha) = T$ *or* $v(\beta) = T$; *and*
5. $v(\alpha \leftrightarrow \beta) = T$ *if and only if* $v(\alpha) = v(\beta)$.

Truth tables are often used even when the formula in question is not broken down all the way into atomic formulas. For example, if α and β are any formulas and we know that α is true but β is false, then the truth of $(\alpha \wedge (\neg\beta))$ can be determined by means of the following table:

α	β	$(\neg\beta)$	$(\alpha \wedge (\neg\beta))$
T	F	T	T

Example 2.2.4 Construct the truth table for $(P \to Q \vee R) \wedge (P \vee \neg Q)$

$$TTTFFFTT$$

Example 2.2.5 Construct the truth table for $P \to (Q \vee R)$

$$TTTFTTTT$$

Definition 2.2.2 *If v is a truth assignment and φ is a formula, we will often say that v satisfies φ if $v(\varphi) = T$. Similarly, if Σ is a set of formulas, we will often say that v satisfies Σ if $v(\sigma) = T$ for every $\sigma \in \Sigma$. We will say that φ (respectively, Σ) is satisfiable if there is some truth assignment which satisfies it.*

Definition 2.2.3 *A formula φ is a tautology if it is satisfied by every truth assignment. i.e $v(\varphi) = T; \forall v$.*

Definition 2.2.4 *A formula ψ is a contradiction if there is no truth assignment which satisfies it. i.e $v(\varphi) = F; \forall v$.*

Example 2.2.6 $\alpha \vee \neg\alpha$ is a tautology.

Example 2.2.7 $\alpha \wedge \neg\alpha$ is a contradiction.

Example 2.2.8 For example, $(A_4 \to A_4)$ is a tautology while $(\neg(A_4 \to A_4))$ is a contradiction, and A_4 is a formula which is neither. One can check whether a given formula is a tautology, contradiction, or neither, by grinding out a complete truth table for it, with a separate line for each possible assignment of truth values to the atomic subformulas of the formula. For $A_3 \to (A_4 \to A_3)$ this gives

A_3	A_4	$A_4 \to A_3$	$A_3 \to (A_4 \to A_3)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

so $A_3 \to (A_4 \to A_3)$ is a tautology.

(iii) $R \to P \to Q \vee R$

$$\begin{aligned} R \to P \to Q \vee R &\equiv \\ &\equiv R \to P \to (Q \vee R) \\ &\equiv R \to (P \to (Q \vee R)) \end{aligned}$$

P	Q	R	$(Q \vee R)$	$(P \to (Q \vee R))$	$(R \to (P \to (Q \vee R)))$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	T	T

Exercise 2.4 Draw truth tables for

1.) $A \to B \wedge C$

2.) $\neg Q \to (P \vee Q)$

3.) $\neg(P \to P \to P)$

4.) $P \leftrightarrow R \wedge S \vee Q$

Exercise 2.5 Prove that

(a)
$$\models P \vee Q \leftrightarrow Q \vee P$$

(b)
$$\models (P \vee Q) \vee (P \to Q)$$

(c)
$$\models A \wedge (A \to B) \to B$$

(d)
$$\models \neg B \wedge (A \to B)\neg A$$

(e)
$$\models (A \wedge B \to C) \to (A \to (B \to C))$$

(f)
$$\models (A \to B) \wedge (A \to C) \leftrightarrow (A \to B \wedge C)$$

(g)
$$\not\models \neg(P \vee Q \leftrightarrow Q \vee P)$$

Example 2.3.2 Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

We shall show this by drawing a single table as below:

p	q	$\neg(q \wedge q)$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The columns for $\neg(p \wedge \neg q)$ and $\neg p \vee \neg q$ are identical, therefore

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Example 2.3.3 Show that

(a)
$$\neg p \leftrightarrow (\neg p \rightarrow q) \not\equiv p \leftrightarrow (p \rightarrow q)$$

(b)
$$p \rightarrow (q \rightarrow p) \equiv \neg p \rightarrow (\neg q \rightarrow \neg p)$$

(c)
$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

(d)
$$(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$$

Example 2.3.4 Prove that $p \rightarrow q$ is not logically equivalent to $(p \wedge q)$

4. *Variables:* $v_0, v_1, v_2, \dots, v_n, \dots$

5. *Equality:* $=$.

6. *A (possibly empty) set of constant symbols.*

7. *For each $k \geq 1$, a (possibly empty) set of k-place function symbols.*

8. *For each $k \geq 1$, a (possibly empty) set of k-place relation (or predicate) symbols.*

The symbols described in parts 1–5 are the logical symbols of $\mathcal{L}$, shared by every first-order language, and the rest are the non-logical symbols of $\mathcal{L}$, which usually depend on what the language's intended use.

Note 2.4.1 *It is possible to define first-order languages without $=$, so $=$ is considered a non-logical symbol by many authors. While such languages have some uses, they are uncommon in ordinary mathematics.*

Observe that any first-order language $\mathcal{L}$ has countably many logical symbols. It may have uncountably many symbols if it has uncountably many non-logical symbols. Unless explicitly stated otherwise, we will assume that every first-order language we encounter has only countably many non-logical symbols. Most of the results we will prove actually hold for countable and uncountable first-order languages alike, but some require heavier machinery to prove for uncountable languages.

Just as in $\mathcal{L}_P$, the parentheses are just punctuation while the connectives, $\neg$ and $\rightarrow$, are intended to express *not* and *if ... then*. However, the rest of the symbols are new and are intended to express ideas that cannot be handled by $\mathcal{L}_P$. The quantifier symbol, $\forall$, is meant to represent *for all*, and is intended to be used with the variable symbols, *e.g.* $\forall v_4$. The constant symbols are meant to be names for particular elements of the structure under discussion. k-place function symbols are meant to name particular functions which map k-tuples of elements of the structure to elements of the structure. k-place relation symbols are intended to name particular k-place relations among elements of the structure.[5] Finally, $=$ is a special binary relation symbol intended to represent equality.

Example 2.4.1 Since the logical symbols are always the same, first-order languages are usually defined by specifying the non-logical symbols. A formal language for elementary number theory like that unofficially described above, call it $\mathcal{L}_{NT}$, can be defined as follows.

- Constant symbols: 0 and 1
- Two 2-place function symbols: $+$ and $\cdot$
- Two binary relation symbols: $<$ and $|$

Each of these symbols is intended to represent the same thing it does in informal mathematical usage: 0 and 1 are intended to be names for the numbers zero and one, $+$ and $\cdot$ names for the operations of addition and multiplications, and $<$ and $|$ names for the relations "less than" and "divides". (Note that we could, in principle, interpret things completely differently – let 0 represent the number forty-one, $+$ the operation of exponentiation, and so on – or even use the

[5] Intuitively, a relation or predicate expresses some (possibly arbitrary) relationship among one or more objects. For example, "n is prime" is a 1-place relation on the natural numbers, $<$ is a 2-place or binary relation on the rationals, and $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{0}$ is a 3-place relation on $\mathbb{R}^3$. Formally, a k-place relation on a set X is just a subset of X^k, *i.e.* the collection of sequences of length k of elements of X for which the relation is true.

Question 2 Solutions

(a) (i) The *formulas* of $\mathcal{L}_P$ are those finite sequences or strings of the symbols which satisfy the following rules:

i. Every atomic formula is a formula.

ii. If α is a formula, then $(\neg\alpha)$ is a formula.

iii. If α and β are formulas, then $(\alpha \to \beta)$ is a formula.

iv. No other sequence of symbols is a formula.

(ii) Suppose φ is a formula of $\mathcal{L}_P$.
The set of *subformulas* of φ, $\mathcal{S}(\varphi)$, is defined as follows.

i. If φ is an atomic formula, then $\mathcal{S}(\varphi) = \{\varphi\}$.

ii. If φ is $(\neg\alpha)$, then $\mathcal{S}(\varphi) = \mathcal{S}(\alpha) \cup \{(\neg\alpha)\}$.

iii. If φ is $(\alpha \to \beta)$, then $\mathcal{S}(\varphi) = \mathcal{S}(\alpha) \cup \mathcal{S}(\beta) \cup \{(\alpha \to \beta)\}$.

(ii1) $S(\neg A_1 \vee A_2) = S(\neg\neg A_1 \to A_2)$

$$\begin{aligned}
&= S(\neg\neg A_1) \cup S(A_2) \cup \{\neg\neg A_1 \to A_2\} \\
&= S(\neg A_1) \cup \{\neg\neg A_1\} \cup \{A_2\} \cup \{\neg\neg A_1 \to A_2\} \\
&= S(A_1) \cup \{\neg A_1\} \cup \{\neg\neg A_1\} \cup \{A_2\} \cup \{\neg\neg A_1 \to A_2\} \\
&= A_1 \cup \{\neg A_1\} \cup \{\neg\neg A_1\} \cup \{A_2\} \cup \{\neg\neg A_1 \to A_2\} \\
&= \{A_1,\ A_2,\ \neg A_1,\ \neg\neg A_1,\ \neg\neg A_1 \to A_2\}
\end{aligned}$$

(ii2) $S(A_1 \to A_2 \to \neg A_3) = S(A_1 \to (A_2 \to \neg A_3))$

$$\begin{aligned}
&= S(A_1) \cup S(A_2 \to \neg A_3) \cup \{A_1 \to (A_2 \to \neg A_3)\} \\
&= S(A_1) \cup S(A_2) \cup S(\neg A_3) \cup \{A_2 \to \neg A_3\} \cup \{A_1 \to (A_2 \to \neg A_3)\} \\
&= \{A_1\} \cup \{A_2\} \cup \{A_3\} \cup \{\neg A_3\} \cup \{A_2 \to \neg A_3\} \cup \{A_1 \to (A_2 \to \neg A_3)\} \\
&= \{A_1,\ A_2,\ A_3,\ \neg A_3,\ A_2 \to \neg A_3,\ A_1 \to (A_2 \to \neg A_3)\}
\end{aligned}$$

(b) (i) If v is a truth assignment and φ is a formula, we will often say that v *satisfies* φ if $v(\varphi) = T$. Similarly, if Σ is a set of formulas, we will often say that v satisfies Σ if $v(\sigma) = T$ for every $\sigma \in \Sigma$. We will say that φ (respectively, Σ) is *satisfiable* if there is some truth assignment which satisfies it.

(ii) A formula φ is a *tautology* if it is satisfied by every truth assignment. i.e $v(\varphi) = T; \forall v$.

(c) Lets use truth tables

(i) $((P \to Q) \wedge (Q \to R)) \to (P \to R)$

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$	$P \to R$	$(P \to Q) \wedge (Q \to R) \to (P \to R$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Benedict xvi resides at the Vatican.

Therefore, Benedict xvi is a pope.

These arguments also have the same form:

All A's are F;

X is F;

Therefore, X is an A.

Arguments with this form are invalid. This is easy to see with the first example. The second example may seem like a good argument because the premises and the conclusion are all true, but note that the conclusion's truth isn't guaranteed by the premises' truth. It could have been possible for the premises to be true and the conclusion false. This argument is invalid, and all invalid arguments are unsound.

While it is accepted by most contemporary logicians that logical validity and invalidity is determined entirely by form, there is some dissent. Consider, for example, the following arguments:

My table is circular.

Therefore, it is not square shaped.

and

Juan is bachelor.

Therefore, he is not married.

These arguments, at least on the surface, have the form:

x is F;

Therefore, x is not G.

Arguments of this form are not valid as a rule. However, it seems clear in these particular cases that it is, in some strong sense, *impossible* for the premises to be true while the conclusion is false. However, many logicians would respond to these complications in various ways. Some might insist- -although this is controversial- -that these arguments actually contain implicit premises such as "Nothing is both circular and square shaped" or "All bachelors are unmarried," which, while themselves necessary truths, nevertheless play a role in the form of these arguments. It might also be suggested, especially with the first argument, that while (even without the additional premise) there is a necessary connection between the premise and the conclusion, the sort of necessity involved is something other than "logical" necessity, and hence that this argument (in the simple form) should not be regarded as logically valid. Lastly, especially with regard to the second example, it might be suggested that because "bachelor" is defined as "adult unmarried male", that the true logical form of the argument is the following universally valid form:

x is F and not G and H;

Therefore, x is not G.

The logical form of a statement is not always as easy to discern as one might expect. For example, statements that seem to have the same surface grammar can nevertheless differ in logical form. Take for example the two statements:

number; let's use this fact and the method of proof by contradiction to prove the statement above.

Given that: r is a rational number, and x is an irrational number.

Show that: $r + x$ is irrational.

Proof: Seeking a contradiction, suppose that $r + x$ is rational.

Since r is rational, $-r$ is also rational; thus the sum of $r + x$ and $-r$ must be a rational number (since the sum of two rational numbers is rational).

Thus $(r + x) + (-r) = x$ is rational; this contradicts our original hypothesis that x is irrational.

Therefore it must not be true that $r + x$ is rational, i.e. we have shown that $r + x$ must be irrational.

Note that in the proof above, we made up some notation; we let r represent an arbitrary rational number, and chose x to represent an arbitrary irrational number. Without doing this the proof would have been very difficult to read and understand. It is also worth noting that if you look up the answer to this problem in the back of the book it says simply "if r and $r + x$ are rational, then $x = (r + x) - r$ is rational." Keep in mind that this is just a hint of the idea of the proof; the snippet in the back of the book is not a complete proof.

Example 3.3.6 If m and n are integers and mn is odd, then m is odd **and** n is odd.
Let m is even **or** n is even

Example 3.3.7 If $m + n$ is irrational, then m is irrational or n is irrational.
Let m is rational **and** n is rational

Exercise 3.1 The cube root of 2 is irrational.

Exercise 3.2 There are no positive integer solutions to the Diophantine equation

$$x^2 - y^2 = 10$$

Exercise 3.3 There is no rational number solution to the equation

$$x^5 + x^4 + x^3 + x^2 + 1 = 0$$

Exercise 3.4 If a^2 is even, the a is even.

Exercise 3.5

(i) $\sqrt{3}$ is irrational.

(ii) $\sqrt{5}$ is irrational.

(iii) If x is odd, then $x + 1$ is even.

Exercise 3.6 Given that $a + b = 10$, $a = 5$, show that $b = 5$.

a guess and go with it. If you guess it's true, then analyze the situation to see why it's true. The analysis might lead to a proof. If you fail to find the proof, you can see why ? then you might discover a counter example. then again, if you fail to get a counterexample, you might begin to suspect again that it is true and formulate reasons why it must be.

Note 3.4.1 As we end the chapter, you should always remember that: The constant emphasis on logic and proofs is what sets mathematics apart from other pursuits.
Precautiously, Many articles and essays are not written to present information clearly and directly; instead they may be written to persuade you to accept a particular viewpoint, to offer an, opinion, to argue for one side of a controversial issue. Consequently, one must recognize and separate factual information from subjective content. Subjective content is any material that involves judgment, feeling, opinion, intuition, or emotion rather than factual information.

Exercise 3.7 Show that prime numbers are infinite.

Example 3.4.3 All students in this class are ugandans

Example 3.4.4 For example, consider the assertion that for **all** integers $n, n > 1$, there are no positive integer solutions for x, y, and z to the equation $x^n + y^n = z^n$. Since for $n = 2, x = 3, y = 4$, and $z = 5$ is a counter example, the assertion must be false. For $n > 2$, however, the statement is true and is known as $Fermat's$ Last Theorem. This famous conjecture remained unresolved until $AndrewWiles$ finally proved it in 1994.

Example 3.4.5 For example, if you're trying to prove the statement "All cheesecakes are baked in Alaska." and you didn't know whether to prove it by contrapositive or contradiction, all I would have to do is bake a cheesecake right in front of you and then you would know that your efforts had been in vain.

Exercise 3.8 Use any appropriate method of prove to check the validity of the statements

(i) All Uganda 2006 presidential candidates were males.

(ii) Every girl put on skirts.

(iii) All Makerere university lecturers are grey- haired.

(iv) An islands is on land.

(v) Cats have nine tails.

(vi) All American presidents have been right-handed.

3.5 Truth Table

Validity of an Argument

An argument arises from propositions called premises and result in a proposition called a conclusion. An argument is valid if and only if the conclusion is true whenever all premises are true. An argument that is valid , if all the premises used in a sequence of reasoning are true then conclusion drawn from them must be true for the argument to be valid. The validity of an argument depends on its form but not on the truth or falsity of the premises. An argument is usually presented in the form:
P_1 : premise
P_2 : premise
Therefore C: conclusion

MEME EMEM
MEEM EMME
MMEE EEMM

It should have been all the 24 ways , but only 6 are right since the others are just a repetition of these.

Example 4.2.6 : In how many ways 4 yellow, 3 red and 2 blue bulbs can be arranged on a string of X-mass tree light with 9 sockets

$$\frac{9!}{(3!\ 4!\ 2!)} = 1260 \text{ ways}$$

Theorem 4.2.5 : The number of ways of partitioning a set of n objects into r cell with n_1 elements in the 1^{st} cell , n_2 in the second cell and so on is

$$\frac{n!}{n_1!\ n_2!\ \cdots\ n_r!}$$

Example 4.2.7 : In how many ways can 7 people be assigned to one triple and two double rooms?

$$\frac{7!}{3!\ 2!\ 2!} = 7 \times 6 \times 5$$

4.3 Combination

The combinations are the best-known element of the four mathematical sets. The lotto drawings are some of the most common representations of the combinations. The random number generation of the arrangements is actually closer to the lotto drawings. The lotto numbers are not drawn in sequential order, but in sequences like 33, 7, 18, 44, 29, 48. The random number generation of the combinations in PermuteCombine.exe is like the lotto draws after sorting the numbers in ascending order.

A combination is an unordered grouping of a set. You might not be interested in arranging (permutation), but in the way you can *select* or *choose* (combination) objects. An example of a combination scenario in which order doesnt matter is a hand of cards: a king, an ace, and a five is the same as an ace, a five, and a king.
Combinations are represented as ${}_nC_r$, where unordered subgroups of size r are selected from a set of size n. Because the order of the elements in a given subgroup doesnt matter, this means that ${}_nC_r$ will be less than ${}_nP_r$. Any one combination can be turned into more than one permutation.

In counting problems where *order* matters, r-permutations are permutation are clearly relevant. Often order is irrelevant , in which case the ability to count sets become important. The number $\begin{pmatrix} n \\ r \end{pmatrix}$ is read " n Choose r" is what is called the number of combinations of n things taken r at a time.

Theorem 4.3.1 For $0 \leq r \leq n$,

$${}_nC_r = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{(n-r)!\ r!}$$

Unemployment,

Environment,

Taxes,

Interest rates,

National defense

And social security

A respondent is to select four categories of concern and rank them by placing the numbers 1,2,3, and 4 after each selected category to indicate the degree of concern with 1 indicating the greatest concern and 4 the least. In how many ways can a respondent reply to the questionnaire? 360

8.) When a coin is tossed, a head(H) or tail(T) may show if a die is rolled, a 1,2,3,4,5 or 6 may show. Suppose a coin is tossed twice and then a die is rolled, and the results are observed , how many different results can occur? 24

9.) Student must take a science course and a humanities course. The science courses that are available are biology, chemistry, physics, computer science and mathematics. The courses available in the humanities course are English, history, speech communication and classics For the two courses the student has how many selections? 20

10.) A person lives in a city A and commutes by automobile to city B. There are 4 roads connecting city A and B

(i) How many routes are possible for a round trip? 16

(ii) How many routes are possible if a different road is to be used for the return trip? 12

11.) At a restaurant a complete dinner consists of an appetizer, an entre, a dessert, and a beverage. For the appetizer, the choices are, soup or juice; for the entre, the choices are chicken, fish, steak or lamb; for the dessert the choices are cherries jubilee, fresh peach cobbler, chocolate truffle cake, a blueberry roly-poly; for the beverage, the choices are coffee, tea or milk. How many complete dinner are possible? 96

12.) In how many ways is it possible to answer the six question multiple choice examination of each question has four and a choices is selected for each question 4096

13.) If a softball league has seven teams, how many different end of the season rankings are possible? Assume that there are no ties. 5040

14.) A coin is tossed four times , how many results are possible of the order of the tosses is considered. 16

15.) (a) In how many ways can five of the seven books be arranged on a book shelf. 2520

(b) In how many ways can all seven books be arranged on a shelf. 5040

16.) The director of research and development for a company has eight computer scientist that are equally qualified to work on project A or project B. In how many ways can director a sign (choose) for scientist to each project. 70

17.) * A manufacturer places five symbol-code as each unit of product. The code consists of a letter followed by 4 numbers, the first of which is not zero. How many codes are possible. 234000

Chapter 5

Boolean Algebra

5.1 Practice Questions

Question 1

(a) State the three outstanding properties of trees.

(b) Define what is meant by the terms:

  (i) binary tree.

  (ii) full Binary tree.

  (iii) m-array tree.

(c) (i) Describe the process of Binary searching.

  (ii) Draw a binary tree involving the names

David, Deborah, Douglas, Dickson, Daniel, Denies and *Diana*

with Douglas as the root of the tree. Explain how you could locate *Deo* from your tree using binary searching.

(d) (i) State one property of prefix codes.

  (ii) Which of the following are prefix codes? and why?

$$a:0\ ,e:1\ ,t:01,\ \ s:001 \text{ or } a:101\ ,e:11\ ,t:001,\ \ s:011,\ \ n:010$$

  (iii) Using the prefix code above, find the word represented by 001011000101.

(a) (i) Define what is meant by

(i1) a set .

(i2) two sets A and B being equal.

(ii) State the two ways of describing a set.

(b) (i) Which of the following sets are equal

$$\begin{array}{llll} A & = \{1, 2, 3\} & B & = \{3, 2, 1, 3\} \\ C & = \{1, 2, 2, 3\} & D & = \{3, 1, 2, 3\} \end{array}$$

State the cardinality of set C.
State the powerset of set A.

(ii) Determine the elements in the set

(i1) $\{n + \frac{1}{n} \ : \ n \ \in \ \{1, 2, 3, 5, 7\}\}$

(i2) $\{n + (-1)^n \ : \ n \ \in \ \mathbb{N}\}$

(c) Let $A = \{a, b, c\}$, write a computer program (or develop an algorithm) that lists all the subsets B of A, where $|B| = 2$.

(d) During freshers' orientation at the main building, two showings of the latest Jennifer Lopez movies were presented. Among the 600 ICT freshers, 80 attended the first movie and 125 attended the second showing, while 450 did not make it to either showing.

(i) How many of the 600 freshers attended twice.

(ii) How many attended either shows.

(iii) What is the probability that a fresher chosen at random attended only one of the movie show.

(e) (i) One of the special sets of numbers declared in any programming language is $\mathbb{Q}$. With a relevant example, define $\mathbb{Q}$.

(ii) Prove that $\sqrt{2}$ is an irrational number ($\mathbb{I}$).

(iii) Solve the equation $3x - 5 = -x$.
According to the classification of numbers , to which group does the x above belong?

(iv) State which of the following statements are true, and which are false.

$$\begin{array}{llll} (a) & \mathbb{Z} \subseteq \mathbb{Q} & (b) & \mathbb{Q} \subseteq \mathbb{R} \\ (c) & \mathbb{Q} \cap \mathbb{R} = \mathbb{Q} & (d) & \mathbb{C} \cup \mathbb{R} = \mathbb{R} \end{array}$$

Question 3 [Methods of Proofs]

(a) (i) State any four methods of proofs.

(ii) Define what is meant for two formulas to be logically equivalent.

(b) Stating the method of proof to use, determine the validity of the statements

(i) All popes reside at the Vatican.
Benedict xvi resides at the Vatican.
Therefore, Benedict xvi is a pope.

(ii) My programming language is C++.
Therefore, it is not Java.

(e) Explaining the terms below :

(i) If v is a truth assignment and φ is a formula, we often say that v *satisfies* φ if $v(\varphi) = T$. Similarly, if Σ is a set of formulas, we often say that v satisfies Σ if $v(\sigma) = T$ for every $\sigma \in \Sigma$. We say that φ (respectively, Σ) is *satisfiable* if there is some truth assignment which satisfies it.

(ii) A formula φ is a *tautology* if it is satisfied by every truth assignment. i.e $v(\varphi) = T; \forall v$.

(f) $(p \wedge (p \to q) \to q)$

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Question 2 [Sets and Numbers]

(a) (i) Defining

(i1) a set is a collection of objects or elements .

(i2) two sets A and B being equal if they have the same elements/members.

(ii) The two ways of describing a set are:

(a) Listing the elements

(b) Builder notation (stating the rule, formular describing the set)

(b) (i)

$$A = B = C = D$$

$|C| = 3$.

the powerset of set A is $\{\{\},\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\ \}$.

(ii) Determining the elements in the set

(i1) $\{2, 2\frac{1}{2}, 3\frac{1}{3}, 5\frac{1}{5}, 7\frac{1}{7}\}$

(i2) $\{0, 2\}$

(c) $B = \{x, y\ : x \neq y\}$ and $x, y \in\ a, b, c$

(d) During freshers' orientation at the main building, two showings of the latest Jennifer Lopez movies were presented. Among the 600 ICT freshers, 80 attended the first movie and 125 attended the second showing, while 450 did not make it to either showing.

(i) How many of the 600 freshers attended twice.
$(125 - x) + x + (180 - x) + 450 = 600$
$205 - x = 150$
$x = 55$

(ii) How many attended either shows.$(125 - x) + x + (180 - x) = 150$

(iii) The probability that a fresher chosen at random attended only one of the movie show.
One movie $= (125 - x) + (180 - x) = 95$
$\frac{95}{600} = \frac{19}{120}$

iii. Atomic formulas: $A_0, A_1, A_2, \ldots, A_n, \ldots$
Atomic formulas are formulas that cannot be broken down into further shorter formulas, e.g He came to class. But 'I will go to the market and buy matooke' is not atomic.

(b) (i)
- $(\alpha \wedge \beta)$ is short for $\neg(\alpha \to \neg\beta)$,
- $(\alpha \vee \beta)$ is short for $(\neg\alpha \to \beta)$, and
- $(\alpha \leftrightarrow \beta)$ is short for $(\alpha \to \beta) \wedge (\beta \to \alpha)$ is short for $\neg((\alpha \to \beta) \to \neg(\beta \to \alpha))$.

(ii) $(((p \vee q) \wedge q) \to p)$

(c) (i) Truth table for $((A \to B) \wedge (A \to C)) \leftrightarrow (A \to (B \wedge C))$

A	B	C	$A \to B$	$A \to C$	$B \wedge C$	$(A \to B) \wedge (A \to C)$	$A \to (B \wedge C)$	$(A \to B) \wedge (A \to C) \leftrightarrow$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	F	F	F	T
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

(ii)

P	Q	R	S	$(S \to R)$	$(P \to (S \to R))$	$Q \to (P \to (S \to R))$
T	F	T	F	T	T	T

(d) Truth table for Implication $(\to)$

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Consider the following compound conditional propositions and interpret the statements.

(i) If Kampala is in Uganda then $3 + 3 = 7$
(ii) If Kampala is in America then $3 + 3 = 6$
(iii) If Kampala is in Uganda then $3 + 3 = 6$
(iv) If Kampala is in America then $3 + 3 = 7$.

The conditional propositions ii, iii and iv are all true. The conditional propositional (i) is however, false since a true statement implies a false statement.

(e) (i) If today is clear, then its not true that its rainy and cold
(ii) If and only if yestarday was cloudy, will today be clear

(f)

A : To access the Internet from campus
M : You are a computer science major
F : You are a freshman

$$A \leftrightarrow (M \wedge \neg F)$$