Equations in Physics

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2. Rotation: $\vec{F}_\alpha = -m\vec{\alpha} \times \vec{r}\,'$
3. Coriolis force: $F_{\rm cor} = -2m\vec{\omega} \times \vec{v}$
4. Centrifugal force: $\vec{F}_{\rm cf} = m\omega^2\vec{r}_n\,' = -\vec{F}_{\rm cp}$; $\vec{F}_{\rm cp} = -\dfrac{mv^2}{r}\vec{e}_r$

1.4.2 Tensor notation

Transformation of the Newtonian equations of motion to $x^\alpha = x^\alpha(x)$ gives:

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt};$$

so

$$\frac{d}{dt}\frac{dx^\alpha}{dt} = \frac{d^2x^\alpha}{dt^2} = \frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt}\right) = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2} + \frac{d\bar{x}^\beta}{dt}\frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right)$$

The chain rule gives:

$$\frac{d}{dt}\frac{\partial x^\alpha}{\partial \bar{x}^\beta} = \frac{\partial}{\partial \bar{x}^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\gamma}{dt} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt}$$

So:

$$\frac{d^2x^\alpha}{dt^2} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt}$$

So the Newtonian equation of motion

$$m\frac{d^2x^\alpha}{dt^2} = F^\alpha$$

will be transformed into:

$$m\left\{\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\right\} = F^\alpha$$

The apparent forces are brought from the origin to the effect side in the way $\Gamma^\alpha_{\beta\gamma}\dfrac{dx^\beta}{dt}\dfrac{dx^\gamma}{dt}$.

1.5 Dynamics of masspoint collections

1.5.1 The center of mass

The velocity w.r.t. the center of mass $\vec{R}$ is given by $\vec{v} - \dot{\vec{R}}$. The coordinates of the center of mass are given by:

$$\vec{r}_{\rm m} = \frac{\sum m_i\vec{r}_i}{\sum m_i}$$

In a 2-particle system, the coordinates of the center of mass are given by:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

With $\vec{r} = \vec{r}_1 - \vec{r}_2$, the kinetic energy becomes: $T = \frac{1}{2}M_{\rm tot}\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$, with the *reduced mass* μ is given by: $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$
The motion within and outside the center of mass can be separated:

$$\dot{\vec{L}}_{\rm outside} = \vec{\tau}_{\rm outside}\ ;\quad \dot{\vec{L}}_{\rm inside} = \vec{\tau}_{\rm inside}$$

$$\vec{p} = m\vec{v}_{\rm m}\ ;\quad \vec{F}_{\rm ext} = m\vec{a}_{\rm m}\ ;\quad \vec{F}_{12} = \mu\vec{u}$$

1. Transfer along length l: $M_{\rm R}=\left(\begin{array}{cc}1&0\\ l/n&1\end{array}\right)$
2. Refraction at a surface with dioptric power D: $M_{\rm T}=\left(\begin{array}{cc}1&-D\\ 0&1\end{array}\right)$

6.4 Aberrations

Lenses usually do not give a perfect image. Some causes are:

1. **Chromatic aberration** is caused by the fact that $n=n(\lambda)$. This can be partially corrected with a lens which is composed of more lenses with different functions $n_i(\lambda)$. Using N lenses makes it possible to obtain the same f for N wavelengths.
2. **Spherical aberration** is caused by second-order effects which are usually ignored; a spherical surface does not make a perfect lens. Incomming rays far from the optical axis will more bent.
3. **Coma** is caused by the fact that the principal planes of a lens are only plane near the principal axis. Further away of the optical axis they are curved. This curvature can be both positive or negative.
4. **Astigmatism**: from each point of an object not on the optical axis the image is an ellipse because the thickness of the lens is not the same everywhere.
5. **Field curvature** can be corrected by the human eye.
6. **Distorsion** gives abberations near the borders of the image. This can be corrected with a combination of positive and negative lenses.

6.5 Reflection and transmission

If an electromagnetic wave hits a transparent medium a part of the wave shall reflect at the same angle as the incident angle, and a part will be refracted at an angle following from Snell's law. It makes a difference whether the $\vec{E}$ field of the wave is $\perp$ or $\parallel$ w.r.t. the surface. When the coefficients of reflection r and transmission t are defined as:

$$r_\parallel\equiv\left(\frac{E_{0r}}{E_{0i}}\right)_\parallel~,~~r_\perp\equiv\left(\frac{E_{0r}}{E_{0i}}\right)_\perp~,~~t_\parallel\equiv\left(\frac{E_{0t}}{E_{0i}}\right)_\parallel~,~~t_\perp\equiv\left(\frac{E_{0t}}{E_{0i}}\right)_\perp$$

where E_{0r} is the reflected amplitude and E_{0t} the transmitted amplitude. Then the Fresnel equations are:

$$r_\parallel=\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}~~,~~r_\perp=\frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)}$$

$$t_\parallel=\frac{2\sin(\theta_t)\cos(\theta_i)}{\sin(\theta_t+\theta_i)\cos(\theta_t-\theta_i)}~~,~~t_\perp=\frac{2\sin(\theta_t)\cos(\theta_i)}{\sin(\theta_t+\theta_i)}$$

The following holds: $t_\perp-r_\perp=1$ and $t_\parallel+r_\parallel=1$. If the coefficient of reflection R and transmission T are defined as (with $\theta_i=\theta_r$):

$$R\equiv\frac{I_r}{I_i}~~~\mbox{and}~~~T\equiv\frac{I_t\cos(\theta_t)}{I_i\cos(\theta_i)}$$

with $I=\langle|\vec{S}|\rangle$ follows: $R+T=1$. Special is the case $r_\perp=0$. This happens if the angle between the reflected and transmitted rays is 90°. From Snell's law then follows: $\tan(\theta_i)=n$. This angle is called *Brewster's angle*. The situation with $r_\parallel=0$ is not possible.

Chapter 7

Statistical physics

7.1 Degrees of freedom

A molecule consisting of n atoms has $s = 3n$ degrees of freedom. There are 3 translational degrees of freedom, a linear molecule has $s = 3n - 5$ vibrational degrees of freedom and a non-linear molecule $s = 3n - 6$. A linear molecule has 2 rotational degrees of freedom and a non-linear molecule 3.

Because vibrational degrees of freedom account for both kinetic and potential energy they count double. So, for linear molecules this results in a total of $s = 6n - 5$. For non-linear molecules this gives $s = 6n - 6$. The average energy of a molecule in thermodynamic equilibrium is $\langle E_{\text{tot}} \rangle = \frac{1}{2} skT$. Each degree of freedom of a molecule has in principle the same energy: the *principle of equipartition.*

The rotational and vibrational energy of a molecule are:

$$W_{\text{rot}} = \frac{\hbar^2}{2I} l(l+1) = Bl(l+1) \ , \quad W_{\text{vib}} = (v + \tfrac{1}{2})\hbar\omega_0$$

The vibrational levels are excited if $kT \approx \hbar\omega$, the rotational levels of a hetronuclear molecule are excited if $kT \approx 2B$. For homonuclear molecules additional selection rules apply so the rotational levels are well coupled if $kT \approx 6B$.

7.2 The energy distribution function

The general shape of the equilibrium velocity distribution function is
$P(v_x, v_y, v_z) dv_x dv_y dv_z = P(v_x) dv_x \cdot P(v_y) dv_y \cdot P(v_z) dv_z$ with

$$P(v_i) dv_i = \frac{1}{\alpha\sqrt{\pi}} \exp\left(-\frac{v_i^2}{\alpha^2} \right) dv_i$$

where $\alpha = \sqrt{2kT/m}$ is the *most probable velocity* of a particle. The average velocity is given by $\langle v \rangle = 2\alpha/\sqrt{\pi}$, and $\langle v^2 \rangle = \frac{3}{2}\alpha^2$. The distribution as a function of the absolute value of the velocity is given by:

$$\frac{dN}{dv} = \frac{4N}{\alpha^3\sqrt{\pi}} \, v^2 \exp\left(-\frac{mv^2}{2kT} \right)$$

The general shape of the energy distribution function then becomes:

$$P(E)dE = \frac{c(s)}{kT} \left(\frac{E}{kT} \right)^{\frac{1}{2}s - 1} \exp\left(-\frac{E}{kT} \right)$$

where $c(s)$ is a normalization constant, given by:

1. Even s: $s = 2l$: $c(s) = \dfrac{1}{(l-1)!}$
2. Odd s: $s = 2l + 1$: $c(s) = \dfrac{2^l}{\sqrt{\pi}(2l-1)!!}$

8.9 Thermodynamic potential

When the number of particles within a system changes this number becomes a third quantity of state. Because addition of matter usually happens at constant p and T, G is the relevant quantity. If a system exists of more components this becomes:

$$dG = -SdT + Vdp + \sum_i \mu_i dn_i$$

where $\mu = \left(\frac{\partial G}{\partial n_i}\right)_{p,T,n_j}$ is called the thermodynamic potential. This is a *partial quantity*. For V holds:

$$V = \sum_{i=1}^{c} n_i \left(\frac{\partial V}{\partial n_i}\right)_{n_j,p,T} := \sum_{i=1}^{c} n_i V_i$$

where V_i is the the partial volume of component i. The following holds:

$$\begin{aligned} V_m &= \sum_i x_i V_i \\ 0 &= \sum_i x_i dV_i \end{aligned}$$

where $x_i = n_i/n$ is the molar fraction of component i. The molar volume of a mixture of two components can be a concave line in a V-x_2 diagram: the mixing contracts the volume.

The thermodynamic potentials are not independent in a multiple-phase system. It can be derived that $\sum_i n_i d\mu_i = -SdT + Vdp$, this gives at constant p and T: $\sum_i x_i d\mu_i = 0$ (Gibbs-Duhmen).

Each component has as much μ's as there are phases. The number of free parameters in a system with c components and p different phases is given by $f = c + 2 - p$.

8.10 Ideal mixtures

For a mixture of n components holds (the index ^{0} is the value for the pure component):

$$U_{\rm mixture} = \sum_i n_i U_i^0 \ \ , \ \ H_{\rm mixture} = \sum_i n_i H_i^0 \ \ , \ \ S_{\rm mixture} = n \sum_i x_i S_i^0 + \Delta S_{\rm mix}$$

where for ideal gases holds: $\Delta S_{\rm mix} = -nR \sum_i x_i \ln(x_i)$.

For the thermodynamic potentials holds: $\mu_i = \mu_i^0 + RT\ln(x_i) < \mu_i^0$. A mixture of two liquids is rarely ideal: this is usually only the case for chemical related components or isotopes. In spite of this holds Raoult's law for the vapor pressure holds for many binary mixtures: $p_i = x_i p_i^0 = y_i p$. Here is x_i the fraction of the ith component in liquid phase and y_i the fraction of the ith component in gas phase.

A solution of one component in another gives rise to an increase in the boiling point $\Delta T_{\rm k}$ and a decrease of the freezing point. $\Delta T_{\rm s}$. For $x_2 \ll 1$ holds:

$$\Delta T_{\rm k} = \frac{RT_{\rm k}^2}{r_{\beta\alpha}} x_2 \ \ , \ \ \Delta T_{\rm s} = -\frac{RT_{\rm s}^2}{r_{\gamma\beta}} x_2$$

with $r_{\beta\alpha}$ the evaporation heat and $r_{\gamma\beta} < 0$ the melting heat. For the *osmotic pressure* Π of a solution holds: $\Pi V_{ml}^0 = x_2 RT$.

8.11 Conditions for equilibrium

When a system evolves towards equilibrium the only changes that are possible are those for which holds: $(dS)_{U,V} \geq 0$ or $(dU)_{S,V} \leq 0$ or $(dH)_{S,p} \leq 0$ or $(dF)_{T,V} \leq 0$ or $(dG)_{T,p} \leq 0$. In equilibrium holds for each component: $\mu_i^\alpha = \mu_i^\beta = \mu_i^\gamma$.

Here, $\nu = \eta/\varrho$ is the *kinematic viscosity*, c is the speed of sound and L is a characteristic length of the system. α follows from the equation for heat transport $\kappa\partial_y T = \alpha\Delta T$ and $a = \kappa/\varrho c$ is the thermal diffusion coefficient.

These numbers can be interpreted as follows:

- Re: (stationary inertial forces)/(viscous forces)
- Sr: (instationary inertial forces)/(stationary inertial forces)
- Fr: (stationary inertial forces)/(gravity)
- Fo: (heat conductance)/(instationary change in enthalpy)
- Pe: (convective heat transport)/(heat conductance)
- Ec: (viscous dissipation)/(convective heat transport)
- Pr and Nu are related to specific materials.

Now, the dimensionless Navier-Stokes equation becomes, with $x' = x/L$, $\vec{v}\,' = \vec{v}/V$, $\text{grad}' = L\text{grad}$, $\nabla'^2 = L^2\nabla^2$ and $t' = t\omega$:

$$\text{Sr}\frac{\partial\vec{v}\,'}{\partial t'} + (\vec{v}\,'\cdot\nabla')\vec{v}\,' = -\text{grad}'p + \frac{\vec{g}}{\text{Fr}} + \frac{\nabla'^2\vec{v}\,'}{\text{Re}}$$

9.5 Tube flows

For tube flows holds: they are laminar if Re< 2300 with as dimension of length the diameter of the tube, and turbulent if Re is larger. For an incompressible laminar flow through a straight, circular tube holds for the velocity profile:

$$v(r) = -\frac{1}{4\eta}\frac{dp}{dx}(R^2 - r^2)$$

For the volume flow holds: $\Phi_V = \int\limits_0^R v(r)2\pi r dr = -\frac{\pi}{8\eta}\frac{dp}{dx}R^4$

The *entrance length* L_e is given by:

1. $500 < \text{Re}_D < 2300$: $L_\text{e}/2R = 0,056\text{Re}_D$
2. $\text{Re} > 2300$: $L_\text{e}/2R \approx 50$

For gas transport at low pressures (Knudsen-gas) holds: $\Phi_V = \frac{4R^3\alpha\sqrt{\pi}}{3}\frac{dp}{dx}$

For flows at a small Re holds: $\nabla p = \eta\nabla^2\vec{v}$ and $\text{div}\vec{v} = 0$. For the total force on a sphere with radius R in a flow then holds: $F = 6\pi\eta Rv$. For large Re holds for the force on a surface A: $F = \frac{1}{2}C_W A\varrho v^2$.

9.6 Potential theory

The *circulation* Γ is defined as: $\Gamma = \oint(\vec{v}\cdot\vec{e}_\text{t})ds = \iint(\text{rot}\vec{v})\cdot\vec{n}d^2A = \iint(\vec{\omega}\cdot\vec{n})d^2A$

For non viscous media, if $p = p(\varrho)$ and all forces are conservative, Kelvin's theorem can be derived:

$$\frac{d\Gamma}{dt} = 0$$

1. $L-S$ coupling: for small atoms is the electrostatic interaction dominant and the state can be characterized by L, S, J, m_J. $J \in \{|L-S|, ..., L+S-1, L+S\}$ and $m_J \in \{-J, ..., J-1, J\}$. The spectroscopic notation for this interaction is: ${}^{2S+1}L_J$. $2S+1$ is the multiplicity of a multiplet.
2. $j-j$ coupling: for larger atoms is the electrostatic interaction smaller then the $L_i \cdot s_i$ interaction of an electron. The state is characterized by $j_i...j_n, J, m_J$ where only the j_i of the not completely filled subshells are to be taken into account.

The energy difference for larger atoms when placed in a magnetic field is: $\Delta E = g\mu_{\rm B} m_J B$ where g is the Landé factor. For a transition between two singlet states the line splits in 3 parts, for $\Delta m_J = -1, 0+1$. This results in the normal Zeeman effect. At higher S the line splits up in more parts: the anomalous Zeeman effect.

Interaction with the spin of the nucleus gives the hyperfine structure.

10.12.4 Selection rules

For the dipole transition matrix elements follows: $p_0 \sim |\langle l_2 m_2 | \vec{E} \cdot \vec{r}\, | l_1 m_1 \rangle|$. Conservation of angular momentum demands that for the transiting electron holds that $\Delta l = \pm 1$.

For an atom where $L-S$ coupling is dominant further holds: $\Delta S = 0$ (but not strict), $\Delta L = 0, \pm 1$, $\Delta J = 0, \pm 1$ except for $J = 0 \rightarrow J = 0$ transitions, $\Delta m_J = 0, \pm 1$, but $\Delta m_J = 0$ is forbidden if $\Delta J = 0$.

For an atom where $j-j$ coupling is dominant further holds: for the jumping electron holds, except $\Delta l = \pm 1$, also: $\Delta j = 0, \pm 1$, and for all other electrons: $\Delta j = 0$. For the total atom holds: $\Delta J = 0, \pm 1$ but no $J = 0 \rightarrow J = 0$ transitions and $\Delta m_J = 0, \pm 1$, but $\Delta m_J = 0$ is forbidden if $\Delta J = 0$.

10.13 Interaction with electromagnetic fields

The Hamiltonian of an electron in an electromagnetic field is given by:

$$H = \frac{1}{2\mu}(\vec{p} + e\vec{A})^2 - eV = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{e}{2\mu}\vec{B} \cdot \vec{L} + \frac{e^2}{2\mu}A^2 - eV$$

where μ is the reduced mass of the system. The term $\sim A^2$ can usually be neglected, except for very strong fields or macroscopic motions. For $\vec{B} = B\vec{e}_z$ it is given by $e^2B^2(x^2+y^2)/8\mu$.

When a gauge transformation $\vec{A}\,' = \vec{A} - \nabla f$, $V' = V + \partial f/\partial t$ is applied on the potentials the wavefunction is also transformed according to $\psi' = \psi {\rm e}^{iqef/\hbar}$ with qe the charge of the particle. Because $f = f(x,t)$, this is called a *local* gauge transformation, in contrast with a *global* gauge transform which can always be applied.

10.14 Perturbation theory

10.14.1 Time-independent perturbation theory

To solve the equation $(H_0 + \lambda H_1)\psi_n = E_n\psi_n$ one has to find the eigenfunctions of $H = H_0 + \lambda H_1$. Suppose that ϕ_n is a complete set eigenfunctions is of the nun-perturbed Hamiltonian H_0: $H_0\phi_n = E_n^0\phi_n$. Because ϕ_n is a complete set holds:

$$\psi_n = N(\lambda)\left\{\phi_n + \sum_{k \neq n} c_{nk}(\lambda)\phi_k\right\}$$

When c_{nk} and E_n are being developed to λ: $c_{nk} = \lambda c_{nk}^{(1)} + \lambda^2 c_{nk}^{(2)} + \cdots$

$E_n = E_n^0 + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$

The resistivity $\eta = E/J$ of a plasma is given by:

$$\eta = \frac{n_e e^2}{m_e \nu_{ei}} = \frac{e^2\sqrt{m_e}\ln(\Lambda_C)}{6\pi\sqrt{3}\varepsilon_0^2(kT_e)^{3/2}}$$

The diffusion coefficient D is defined via the flux Γ by $\vec{\Gamma} = n\vec{v}_{\rm diff} = -D\nabla n$. The equation of continuity is $\partial_t n + \nabla(nv_{\rm diff}) = 0 \Rightarrow \partial_t n = D\nabla^2 n$. One finds that $D = \frac{1}{3}\lambda_v v$. A rough estimate gives $\tau_D = L_p/D = L_p^2\tau_c/\lambda_v^2$. For magnetized plasma's λ_v must be replaced with the cyclotron radius. In electrical fields also holds $\vec{J} = ne\mu\vec{E} = e(n_e\mu_e + n_i\mu_i)\vec{E}$ with $\mu = e/m\nu_c$ the mobility of the particles. The Einstein ratio is:

$$\frac{D}{\mu} = \frac{kT}{e}$$

Because a plasma is electrical neutral electrons and ions are strongly coupled and they don't diffuse independent. The *coefficient of ambipolar diffusion* $D_{\rm amb}$ is defined by $\vec{\Gamma} = \vec{\Gamma}_i = \vec{\Gamma}_e = -D_{\rm amb}\nabla n_{e,i}$. From this follows that

$$D_{\rm amb} = \frac{kT_e/e - kT_i/e}{1/\mu_e - 1/\mu_i} \approx \frac{kT_e\mu_i}{e}$$

In an external magnetic field B_0 particles will move in spiral orbits with *cyclotron radius* $\rho = mv/eB_0$ and with cyclotron frequency $\Omega = B_0 e/m$. The spiralized orbit is disturbed by collisions. A plasma is called *magnetized* if $\lambda_v > \rho_{e,i}$. So the electrons are magnetized if

$$\frac{\rho_e}{\lambda_{ee}} = \frac{\sqrt{m_e}e^3 n_e\ln(\Lambda_C)}{6\pi\sqrt{3}\varepsilon_0^2(kT_e)^{3/2}B_0} < 1$$

Magnetization of only the electrons is sufficient to confine the plasma reasonable because they are coupled to the ions by charge neutrality. In case of magnetic confinement holds: $\nabla p = \vec{J}\times\vec{B}$. Combined with the two stationary Maxwell equations for the B-field these form the ideal magneto-hydrodynamic equations. For a uniform B-field holds: $p = nkT = B^2/2\mu_0$.

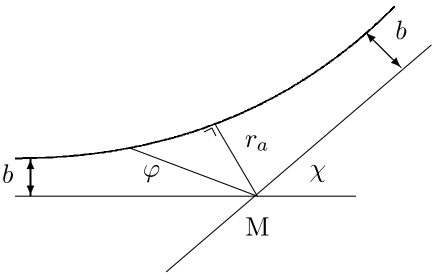
If both magnetic and electric fields are present electrons and ions will move in the same direction. If $\vec{E} = E_r\vec{e}_r + E_z\vec{e}_z$ and $\vec{B} = B_z\vec{e}_z$ the $\vec{E}\times\vec{B}$ drift results in a velocity $\vec{u} = (\vec{E}\times\vec{B})/B^2$ and the velocity in the r,φ plane is $\dot{r}(r,\varphi,t) = \vec{u} + \dot{\vec{\rho}}(t)$.

11.3 Elastic collisions

11.3.1 General

The scattering angle of a particle in interaction with another particle, as shown in the figure at the right is:

$$\chi = \pi - 2b\int\limits_{r_a}^{\infty}\frac{dr}{r^2\sqrt{1 - \dfrac{b^2}{r^2} - \dfrac{W(r)}{E_0}}}$$



Particles with an impact parameter between b and $b + db$, going through a ring with $d\sigma = 2\pi b db$ leave the scattering area at a solid angle $d\Omega = 2\pi\sin(\chi)d\chi$. The *differential cross section* is then defined as:

$$I(\Omega) = \left|\frac{d\sigma}{d\Omega}\right| = \frac{b}{\sin(\chi)}\frac{\partial b}{\partial\chi}$$

For a potential energy $W(r) = kr^{-n}$ follows: $I(\Omega, v) \sim v^{-4/n}$.

For low energies, $\mathcal{O}(1 \text{ eV})$, σ has a *Ramsauer minimum*. It arises from the interference of matter waves behind the object. $I(\Omega)$ for angles $0 < \chi < \lambda/4$ is larger than the classical value.

11.3.2 The Coulomb interaction

For the Coulomb interaction holds: $2b_0 = q_1q_2/2\pi\varepsilon_0 mv_0^2$, so $W(r) = 2b_0/r$. This gives $b = b_0\cot(\frac{1}{2}\chi)$ and

$$I(\Omega = \frac{b}{\sin(\chi)}\frac{\partial b}{\partial\chi} = \frac{b_0^2}{4\sin^2(\frac{1}{2}\chi)}$$

Because the influence of a particle vanishes at $r = \lambda_{\rm D}$ holds: $\sigma = \pi(\lambda_{\rm D}^2 - b_0^2)$. Because $dp = d(mv) = mv_0(1-\cos\chi)$ a cross section related to momentum transfer $\sigma_{\rm m}$ is given by:

$$\sigma_{\rm m} = \int(1-\cos\chi)I(\Omega)d\Omega = 4\pi b_0^2\ln\left(\frac{1}{\sin(\frac{1}{2}\chi_{\rm min})}\right) = 4\pi b_0^2\ln\left(\frac{\lambda_{\rm D}}{b_0}\right) := 4\pi b_0^2\ln(\Lambda_{\rm C}) \sim \frac{\ln(v^4)}{v^4}$$

where $\ln(\Lambda_{\rm C})$ is the *Coulomb-logarithm*. For this quantity holds: $\Lambda_{\rm C} = \lambda_{\rm D}/b_0 = 9n(\lambda_{\rm D})$.

11.3.3 The induced dipole interaction

The induced dipole interaction, with $\vec{p} = \alpha\vec{E}$, gives a potential V and an energy W in a dipole field given by:

$$V(r) = \frac{\vec{p}\cdot\vec{e}_r}{4\pi\varepsilon_0 r^2} \quad , \quad W(r) = -\frac{|e|p}{8\pi\varepsilon_0 r^2} = -\frac{\alpha e^2}{2(4\pi\varepsilon_0)^2 r^4}$$

with $b_a = \sqrt[4]{\frac{2e^2\alpha}{(4\pi\varepsilon_0)^2\frac{1}{2}mv_0^2}}$ holds: $\chi = \pi - 2b\int\limits_{r_a}^{\infty}\frac{dr}{r^2\sqrt{1-\frac{b^2}{r^2}+\frac{b_a^4}{4r^4}}}$

If $b \geq b_a$ the charge would hit the atom. Repulsing nuclear forces prevent this to happen. If the scattering angle is many times 2π it is called capture. The cross section for capture $\sigma_{\rm orb} = \pi b_a^2$ is called the Langevin limit, and is a lowest estimate for the total cross section.

11.3.4 The center of mass system

If collisions of two particles with masses m_1 and m_2 who scatter in the center of mass system under an angle χ are compared with the scattering under an angle θ in the laboratory system holds:

$$\tan(\theta) = \frac{m_2\sin(\chi)}{m_1 + m_2\cos(\chi)}$$

The energy loss ΔE of the incoming particle is given by:

$$\frac{\Delta E}{E} = \frac{\frac{1}{2}m_2v_2^2}{\frac{1}{2}m_1v_1^2} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos(\chi))$$

11.3.5 Scattering of light at free electrons

Scattering of light at free electrons is called Thomson scattering. The scattering is free of collective effects if $k\lambda_{\rm D} \ll 1$. The cross section $\sigma = 6,65\cdot 10^{-29}{\rm m}^2$ and

$$\frac{\Delta f}{f} = \frac{2v}{c}\sin(\tfrac{1}{2}\chi)$$

This gives for the scattered energy $E_{\rm scat} = N\lambda_0^4/(\lambda^2-\lambda_0^2)^2$. If relativistic effects become important, this limit of Compton scattering (which is given by $\lambda' - \lambda = \lambda_{\rm C}(1-\cos\chi)$ with $\lambda_{\rm C} = h/mc$) can not be used any more.

11.4 Thermodynamic equilibrium and reversibility

For a plasma in equilibrium holds Planck's radiation law and the Maxwellian velocity distribution:

$$\rho(\nu,T)d\nu=\frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1}d\nu\quad,\quad N(E,T)dE=\frac{2\pi n}{(\pi kT)^{3/2}}\sqrt{E}\exp\left(-\frac{E}{kT}\right)dE$$

"Detailed balancing" means that the number of reactions in one direction equals the number of reactions in the opposite direction because both processes have equal probability if one corrects for the used phase space. For the reaction

$$\sum_{\rm forward}{\rm X}_{\rm forward}\rightleftarrows\sum_{\rm back}{\rm X}_{\rm back}$$

holds in a plasma in equilibrium *microscopic* reversibility:

$$\prod_{\rm forward}\hat{\eta}_{\rm forward}=\prod_{\rm back}\hat{\eta}_{\rm back}$$

If the velocity distribution is Maxwellian, this gives:

$$\hat{\eta}_x=\frac{n_x}{g_x}\frac{h^3}{(2\pi m_xkT)^{3/2}}{\rm e}^{-E_{\rm kin}/kT}$$

where g is the statistical weight of the state and $n/g:=\eta$. For electrons holds $g=2$, for excited states usually holds $g=2j+1=2n^2$.

With this one finds for the Boltzmann balance: ${\rm X}_p+{\rm e}^-\rightleftarrows{\rm X}_1+{\rm e}^-+(E_{1p})$:

$$\frac{n_p}{n_1}=\frac{g_p}{g_1}\exp\left(-\frac{E_p-E_1}{kT_{\rm e}}\right)$$

And for the Saha balance: ${\rm X}_p+{\rm e}^-+(E_{pi})\rightleftarrows{\rm X}_1^++2{\rm e}^-$:

$$\frac{n_p^{\rm S}}{g_p}=\frac{n_1^+}{g_1^+}\frac{n_{\rm e}}{g_{\rm e}}\frac{h^3}{(2\pi m_{\rm e}kT_{\rm e})^{3/2}}\exp\left(\frac{E_{pi}}{kT_{\rm e}}\right)$$

Because the number of particles on the left-hand side and right-hand side of the equation is different, a factor $g/V_{\rm e}$ remains. This factor causes the Saha-jump.

From microscopic reversibility one can derive that for the rate coefficients $K(p,q,T):=\langle\sigma v\rangle_{pq}$ holds:

$$K(q,p,T)=\frac{g_p}{g_q}K(p,q,T)\exp\left(\frac{\Delta E_{pq}}{kT}\right)$$

11.5 Inelastic collisions

11.5.1 Types of collisions

The kinetic energy can be split in a part *of* and a part *in* the center of mass system. The energy *in* the center of mass system is available for reactions. This energy is given by

$$E=\frac{m_1m_2(v_1-v_2)^2}{2(m_1+m_2)}$$

Some types of inelastic collisions important for plasma physics are:

1. Excitation: ${\rm A}_p+{\rm e}^-\rightleftarrows{\rm A}_q+{\rm e}^-$
2. Decay: ${\rm A}_q\rightleftarrows{\rm A}_p+hf$

12.4 Magnetic field in the solid state

12.4.1 Dielectrics

The quantummechanical origin of diamagnetism is the Larmorprecession of the spin of the electron. Starting with a circular electron orbit in an atom with two electrons, there is a Coulomb force $F_{\rm c}$ and a magnetic force on each electron. If the magnetic part of the force is not strong enough to significant deform the orbit holds:

$$\omega^2 = \frac{F_{\rm c}(r)}{mr} \pm \frac{eB}{m}\omega = \omega_0^2 \pm \frac{eB}{m}(\omega_0 + \delta) \Rightarrow \omega = \sqrt{\left(\omega_0 \pm \frac{eB}{2m}\right)^2 + \cdots} \approx \omega_0 \pm \frac{eB}{2m} = \omega_0 \pm \omega_{\rm L}$$

Here, $\omega_{\rm L}$ is the *Larmor frequency*. One electron is accelerated, the other decelerated. So there is a net circular current which results in a magnetic moment $\vec{\mu}$. The circular current is given by $I = -Ze\omega_{\rm L}/2\pi$, and $\langle\mu\rangle = IA = I\pi\left\langle\rho^2\right\rangle = \frac{2}{3}I\pi\left\langle r^2\right\rangle$. If N is the number of atoms in the crystal follows for the susceptibility, with $\vec{M} = \vec{\mu}N$:

$$\chi = \frac{\mu_0 M}{B} = -\frac{\mu_0 N Z e^2}{6m}\left\langle r^2\right\rangle$$

12.4.2 Paramagnetism

Starting with the splitting of energy levels in a weak magnetic field: $\Delta U_m - \vec{\mu}\cdot\vec{B} = m_J g\mu_{\rm B}B$, and with a distribution $f_m \sim \exp(-\Delta U_m/kT)$, one finds for the average magnetic moment $\langle\mu\rangle = \sum f_m\mu/\sum f_m$. After linearization, and because $\sum m_J = 0$, $\sum J = 2J+1$ and $\sum m_J^2 = \frac{2}{3}J(J+1)(J+\frac{1}{2})$ it follows that:

$$\chi_p = \frac{\mu_0 M}{B} = \frac{\mu_0 N\langle\mu\rangle}{B} = \frac{\mu_0 J(J+1)g^2\mu_{\rm B}^2 N}{3kT}$$

This is the *Curie law*, $\chi_p = C/T$.

12.4.3 Ferromagnetism

A ferromagnet behaves like a paramagnet above a critical temperature $T_{\rm c}$. To describe ferromagnetism a field B_E parallel with M is postulated: $\vec{B}_E = \lambda\mu_0\vec{M}$. The treatment is further analogous with paramagnetism:

$$\mu_0 M = \chi_p(B_{\rm a} + B_E) = \chi_p(B_{\rm a} + \lambda\mu_0 M) = \mu_0\left(1 - \lambda\frac{C}{T}\right)M$$

From this follows for a ferromagnet: $\chi_F = \dfrac{\mu_0 M}{B_{\rm a}} = \dfrac{C}{T - T_{\rm c}}$ this is *Weiss-Curie's law.*

If B_E is estimated this way it results in values of about 1000 T. This is clearly unrealistic and suggests an other mechanism. A quantummechanical approach from Heisenberg postulates an interaction between two neighbor atoms: $U = -2J\vec{S}_i\cdot\vec{S}_j \equiv -\vec{\mu}\cdot\vec{B}_E$. J is an overlap integral given by: $J = 3kT_{\rm c}/2zS(S+1)$, with z the number of neighbors. A distinction between 2 cases can now be made:

1. $J > 0$: S_i and S_j become parallel: the material is a ferromagnet.
2. $J < 0$: S_i and S_j become antiparallel: the material is an antiferromagnet.

Heisenberg's theory predicts quantized spin waves: magnons. Starting with a model with only nearest neighbor interaction one can write:

$$U = -2J\vec{S}_p\cdot(\vec{S}_{p-1} + \vec{S}_{p+1}) \approx \vec{\mu}_p\cdot\vec{B}_p \quad \text{with} \quad \vec{B}_p = \frac{-2J}{g\mu_{\rm B}}(\vec{S}_{p-1} + \vec{S}_{p+1})$$

Chapter 13

Theory of groups

13.1 Introduction

13.1.1 Definition of a group

$\mathcal{G}$ is a group for the operation $\bullet$ if:

1. $\forall_{A,B\in\mathcal{G}} \Rightarrow A\bullet B\in\mathcal{G}$: $\mathcal{G}$ is *closed.*

2. $\forall_{A,B,C\in\mathcal{G}} \Rightarrow (A\bullet B)\bullet C = A\bullet(B\bullet C)$: $\mathcal{G}$ the *associative law.*

3. $\exists_{E\in\mathcal{G}}$ so that $\forall_{A\in\mathcal{G}} A\bullet E = E\bullet A = A$: $\mathcal{G}$ has a *unit element.*

4. $\forall_{A\in\mathcal{G}}\exists_{A^{-1}\in\mathcal{G}}$ z.d.d. $A\bullet A^{-1} = E$: Each element in $\mathcal{G}$ has an *inverse.*

If also holds:
5. $\forall_{A,B\in\mathcal{G}} \Rightarrow A\bullet B = B\bullet A$ the group is called *Abelian* or *commutative.*

13.1.2 The Cayley table

Each element arises only once in each row and column of the Cayley or multiplication table: because $EA_i = A_k^{-1}(A_kA_i) = A_i$ each A_i appears once. There are h positions in each row and column when there are h elements in the group so each elements appears only once.

13.1.3 Conjugated elements, subgroups and classes

B is *conjugate* with A if $\exists_{X\in\mathcal{G}}$ such that $B = XAX^{-1}$. Then A is also conjugate with B because $B = (X^{-1})A(X^{-1})^{-1}$.
If B and C are conjugate with A, B is also conjugate with C.

A *subgroup* is a subset of $\mathcal{G}$ which is also a group for the same operation.

A *conjugacy class* is the maximum collection of conjugated elements. Each group can be split up in conjugacy classes. Some theorems:

- All classes are completely disjoint.
- E is a class itself: for each other element in this class would hold: $A = XEX^{-1} = E$.
- E is the only class which is also a subgroup because all other classes have no unit element.
- In an Abelian group each element is a separate class.

The physical interpretation of classes: elements of a group are usually symmetry operations who map a symmetrical object on itself. Elements of one class are then the same kind of operations. The opposite need not to be true.

as a linear combination of the 3 *Pauli-matrices* σ_i. So these matrices are a choice for the operators of SU(2). One can write: SU(2)=$\{\exp(-\frac{1}{2}i\vec{\sigma}\cdot\vec{\Theta})\}$.

Abstractly, one can consider an isomorphic group where only the commutation rules are considered known about the operators T_i: $[T_1, T_2] = iT_3$, etc.

In elementary particle physics the T_i can be interpreted e.g. as the *isospin* operators. Elementary particles can be classified in isospin-multiplets, this are the irreducible representations of SU(2). The classification is:

1. The isospin-singlet ≡ the identical representation: $\mathrm{e}^{-i\vec{T}\cdot\vec{\Theta}} = 1 \Rightarrow T_i = 0$
2. The isospin-doublet ≡ the faithful representation of SU(2) on 2×2 matrices.

The group SU(3) has 8 free parameters. (The group SU(N) has $N^2 - 1$ free parameters). The Hermitian, traceless operators are 3 SU(2)-subgroups in the $\vec{e}_1\vec{e}_2$, $\vec{e}_1\vec{e}_3$ and the $\vec{e}_2\vec{e}_3$ plane. This gives 9 matrices, who are not all 9 linear independent. By taking a linear combination one gets 8 matrices.

In the Lagrange density for the color force one has to substitute $\displaystyle\frac{\partial}{\partial x} \rightarrow \frac{D}{Dx} := \frac{\partial}{\partial x} - \sum_{i=1}^{8} T_i A_x^i$

The terms of 3rd and 4th power in A show that the color field interacts with itself.

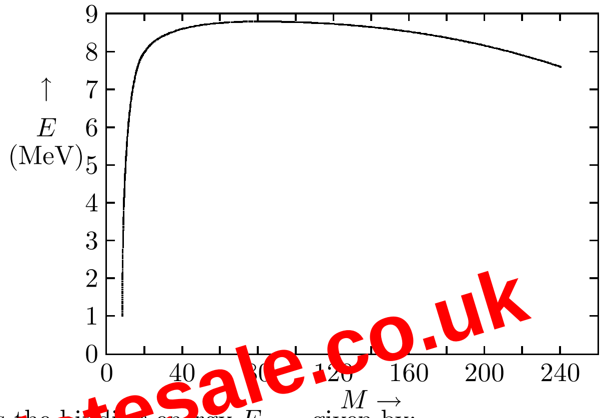
Chapter 14

Nuclear physics

14.1 Nuclear forces

The mass of a nucleus is given by:

$$M_{\rm nucl} = Zm_{\rm p} + Nm_{\rm n} - E_{\rm bind}/c^2$$



The binding energy per nucleon is given in the figure at the right. The top is at $^{56}_{26}$Fe, the most stable nucleus. With the constants

a_1 = 15,760 MeV
a_2 = 17,810 MeV
a_3 = 0,711 MeV
a_4 = 23,702 MeV
a_5 = 34,000 MeV

is in the *droplet* or *collective model* of the nucleus the binding energy $E_{\rm bind}$ given by:

$$\frac{E_{\rm bind}}{c^2} = a_1 A - a_2 A^{2/3} - a_3 \frac{Z(Z-1)}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} + \epsilon a_5 A^{-3/4}$$

These terms arise from:

1. a_1: Binding energy of the strong nuclear force, approximately $\sim A$.
2. a_2: Surface correction: the nucleons near the surface are less bound.
3. a_3: Coulomb repulsion between the protons.
4. a_4: Asymmetry term: a surplus of protons or neutrons has a lower binding energy.
5. a_5: Pair off effect: nuclei with an even number of protons or neutrons are more stable because groups of two protons or neutrons have a lower energy. The following holds:

   $\underline{Z \text{ even}, N \text{ even}}$: $\epsilon = +1$, $\underline{Z \text{ odd}, N \text{ odd}}$: $\epsilon = -1$.
   $\underline{Z \text{ even}, N \text{ odd}}$: $\epsilon = 0$, $\underline{Z \text{ odd}, N \text{ even}}$: $\epsilon = 0$.

The Yukawa potential can be derived if the nuclear force can, in first approximation, be considered an exchange of virtual pions:

$$U(r) = -\frac{W_0 r_0}{r} \exp\left(-\frac{r}{r_0}\right)$$

With $\Delta E \cdot \Delta t \approx \hbar$, $r_0 = c\Delta t$ and $E_\gamma = m_0 c^2$ holds: $r_0 = \hbar / m_0 c$.

In the shell model of the nucleus one assumes that a nucleon moves in an average field of other nucleons. Further, there is a contribution of the spin-orbit coupling $\sim \vec{L} \cdot \vec{S}$: $\Delta V_{ls} = \frac{1}{2}(2l+1)\hbar\omega$. So each level (n, l) niveau is split in two, with $j = l \pm \frac{1}{2}$, where the state with $j = l + \frac{1}{2}$ has the lowest energy. This is just the opposite for electrons, which is an indication that the $L - S$ interaction is not electromagnetical. The energy of a 3-dimensional harmonic oscillator is $E = (N + \frac{3}{2})\hbar\omega$. $N = n_x + n_y + n_z = 2(n-1) + l$ where $n \geq 1$ is the main oscillator number. Because $-l \leq m \leq l$

- Expand all fields to creation and annihilation operators,
- Keep all terms who have no annihilation operators, or in which they are at the right of the creation operators,
- In all other terms interchange the factors so that the annihilation operators go to the right. By an interchange of two fermion operators add a $-$ sign, by interchange of two boson operators not. Assume hereby that all commutators are 0.

15.8 Quantization of the electromagnetic field

Starting with the Lagrange density $\mathcal{L} = -\frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu}$

follows for the field operators $A(x)$:

$$A(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_k}}\sum_{m=1}^{4}\left(a_m(\vec{k})\epsilon^m(\vec{k})\mathrm{e}^{ikx} + a^\dagger(\vec{k})\epsilon^m(\vec{k})^*\mathrm{e}^{-ikx}\right)$$

The operators obey $[a_m(\vec{k}), a^\dagger_{m'}(\vec{k})] = \delta_{mm'}\delta_{kk'}$. All other commutators are 0. m gives the polarization direction of the photon: $m = 1, 2$ gives transversal polarized, $m = 3$ longitudinal polarized and $m = 4$ timelike polarized photons. Further holds:

$$[A_\mu(x), A_\nu(x')] = i\delta_{\mu\nu}D(x - x') \quad \text{with} \quad D(y) = \Delta(y)|_{m=0}$$

In spite of the fact that $A_4 = iV$ is imaginary in the classical case, A_4 is still defined to be hermitian because otherwise the sign of the energy becomes incorrect. By changing the definition of the inner product in configuration space the expectation value for $A_{1,2,3}(x) \in I\!R$ and for $A_4(x)$ imaginary.

If the potentials satisfy the Lorentz gauge condition $\partial_\mu A_\mu = 0$ the E and B operators derived from these potentials will satisfy the Maxwell equations. However, this gives problems with the commutation rules. There is now demanded that only those states are permitted for which holds

$$\frac{\partial A^+_\mu}{\partial x_\mu}|\Phi\rangle = 0$$

This results in: $$\left\langle\frac{\partial A_\mu}{\partial x_\mu}\right\rangle = 0.$$

From this follows that $(a_3(\vec{k}) - a_4(\vec{k}))|\Phi\rangle = 0$. With a local gauge transformation one obtains $N_3(\vec{k}) = 0$ and $N_4(\vec{k}) = 0$. However, this only applies for free EM-fields: in intermediary states in interactions there can exist longitudinal and timelike photons. These photons are also responsible for the stationary Coulomb potential.

15.9 Interacting fields and the S-matrix

The S(scattering)-matrix gives a relation between the initial and final states for an interaction: $|\Phi(\infty)\rangle = S|\Phi(-\infty)\rangle$. If the Schrödinger equation is integrated:

$$|\Phi(t)\rangle = |\Phi(-\infty)\rangle - i\int\limits_{-\infty}^{t} H_{\text{int}}(t_1)|\Phi(t_1)\rangle dt_1$$

and perturbation theory is applied one finds that:

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int T\left\{\mathcal{H}_{\text{int}}(x_1)\cdots\mathcal{H}_{\text{int}}(x_n)\right\}d^4x_1\cdots d^4x_n \equiv \sum_{n=0}^{\infty}S^{(n)}$$

Here, the T-operator means a *time-ordened product*: the terms in such a product must be oredned in increasing time order from the right to the left so that the earliest terms work first. The S-matrix is then given by: $S_{ij} = \langle\Phi_i|S|\Phi_j\rangle = \langle\Phi_i|\Phi(\infty)\rangle$.

The interaction Hamilton density for the interaction between the electromagnetic and the electron-positron field is: $\mathcal{H}_{\rm int}(x) = -J_\mu(x)A_\mu(x) = ieN(\overline{\psi}\gamma_\mu\psi A_\mu)$

When this is expanded as: $\mathcal{H}_{\rm int} = ieN\left((\overline{\psi^+} + \overline{\psi^-})\gamma_\mu(\psi^+ + \psi^-)(A^+_\mu + A^-_\mu)\right)$

eight terms appear. Each term corresponds with a possible process. The term $ie\overline{\psi^+}\gamma_\mu\psi^+A^-_\mu$ working on $|\Phi\rangle$ gives transitions where A^-_μ creates a photon, ψ^+ annihilates an electron and $\overline{\psi^+}$ annihilates a positron. Only terms with the correct number of particles in the initial and final state contribute to a matrixelement $\langle\Phi_i|S|\Phi_j\rangle$. Further the factors in $\mathcal{H}_{\rm int}$ can create and thereafter annihilate particles: the *virtual particles*.

The expressions for $S^{(n)}$ contain time-ordened products of normal products. This can be written as a sum of normal products. The appearing operators describe the minimal changes necessary to change the initial state in the final state. The effects of the virtual particles are described by the (anti)commutator functions. Some time-ordened products are:

$$\begin{array}{rcl}
T\left\{\Phi(x)\Phi(y)\right\} &=& N\left\{\Phi(x)\Phi(y)\right\} + \frac{1}{2}\Delta^{\rm F}(x-y)\\
T\left\{\psi_\alpha(x)\overline{\psi_\beta(y)}\right\} &=& N\left\{\psi_\alpha(x)\overline{\psi_\beta(y)}\right\} - \frac{1}{2}S^{\rm F}_{\alpha\beta}(x-y)\\
T\left\{A_\mu(x)A_\nu(y)\right\} &=& N\left\{A_\mu(x)A_\nu(y)\right\} + \frac{1}{2}\delta_{\mu\nu}D^{\rm F}_{\mu\nu}(x-y)
\end{array}$$

Here, $S^{\rm F}(x) = (\gamma_\mu\partial_\mu - M)\Delta^{\rm F}(x)$, $D^{\rm F}(x) = \Delta^{\rm F}(x)|_{m=0}$ and

$$\Delta^{\rm F}(x) = \left\{\begin{array}{l}
\displaystyle\frac{1}{(2\pi)^3}\int\frac{{\rm e}^{ikx}}{\omega_{\vec{k}}}d^3k \ \ \text{if } x_0 > 0\\[2ex]
\displaystyle\frac{1}{(2\pi)^3}\int\frac{{\rm e}^{-ikx}}{\omega_{\vec{k}}}d^3k \ \ \text{if } x_0 < 0
\end{array}\right.$$

The term $\frac{1}{2}\Delta^{\rm F}(x-y)$ is called the contraction of $\Phi(x)$ and $\Phi(y)$, and is the expectation value of the time-ordened product in the vacuum state. Wick's theorem gives an expression for the time-ordened product of an arbitrary number of field operators. The graphical representation of these processes are called *Feynman diagrams*. In the x-representation each diagram describes a number of processes. The contraction functions can also be written as:

$$\Delta^{\rm F}(x) = \lim_{\epsilon\to 0}\frac{-2i}{(2\pi)^4}\int\frac{{\rm e}^{ikx}}{k^2+m^2-i\epsilon}d^4k \quad\text{and}\quad S^{\rm F}(x) = \lim_{\epsilon\to 0}\frac{-2i}{(2\pi)^4}\int{\rm e}^{ipx}\frac{i\gamma_\mu p_\mu - M}{p^2+M^2-i\epsilon}d^4p$$

In the expressions for $S^{(2)}$ this gives rise to terms $\delta(p+k-p'-k')$. This means that energy and momentum is conserved. However, virtual particles do not obey the relation between energy and momentum.

15.10 Divergences and renormalization

It turns out that higher order contribute infinitely much because only the sum $p+k$ of the four-momentum of the virtual particles is fixed. An integration over one of both becomes ∞. In the x-representation this can be understood because the product of two functions containing δ-like singularities is not well defined. This is solved by discounting all divergent diagrams in a renormalization of e and M. It is assumed that an electron, if there would not be an electromagnetical field, would have a mass M_0 and a charge e_0 unequal to the observed mass M and charge e. In the Hamilton and Lagrange density of the free electron-positron field appears M_0. So this gives, with $M = M_0 + \Delta M$:

$$\mathcal{L}_{\rm e-p}(x) = -\overline{\psi(x)}(\gamma_\mu\partial_\mu + M_0)\psi(x) = -\overline{\psi(x)}(\gamma_\mu\partial_\mu + M)\psi(x) + \Delta M\overline{\psi(x)}\psi(x)$$

and $\mathcal{H}_{\rm int} = ieN(\overline{\psi}\gamma_\mu\psi A_\mu) - i\Delta eN(\overline{\psi}\gamma_\mu\psi A_\mu)$.

where μ is the average molecular mass, usually well approximated by:

$$\mu = \frac{\varrho}{n m_{\rm H}} = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

where X is the mass fraction of H is, Y the mass fraction of He and Z the mass fraction of the other elements. Further holds:

$$\kappa(r) = f(\varrho(r), T(r), \text{composition}) \quad\text{and}\quad \varepsilon(r) = g(\varrho(r), T(r), \text{composition})$$

Convection will occur when the star meets the Schwartzschild criterium:

$$\left(\frac{dT}{dr}\right)_{\rm conv} < \left(\frac{dT}{dr}\right)_{\rm stral}$$

Otherwise the energy transfer shall be by radiation. For stars in quasi-hydrostatic equilibrium hold the approximations $r = \frac{1}{2}R$, $M(r) = \frac{1}{2}M$, $dM/dr = M/R$, $\kappa \sim \varrho$ and $\varepsilon \sim \varrho T^\mu$ (this last assumption is only valid for stars on the main sequence). For pp-chains holds $\mu \approx 5$ and for the CNO chaines holds $\mu = 12$ tot 18. It can be derived that $L \sim M^3$: the *mass-brightness relation*. Further holds: $L \sim R^4 \sim T_{\rm eff}^8$. This results in the equation of the main sequence in the Hertzsprung-Russel diagram:

$$^{10}\log(L) = 8 \cdot {}^{10}\log(T_{\rm eff}) + \text{constant}$$

16.5 Energy production in stars

The net reaction from which most stars gain their energy is: $4\,{}^1\text{H} \rightarrow {}^4\text{He} + 2\text{e}^+ + 2\nu_{\rm e} + \gamma$.
This reaction produces 26,72 MeV. Two reaction chains are responsible for this reaction. The slowest, speed-limiting reaction is shown in boldface. The energy between brackets is the energy cried away by the neutrino.

1. The proton-proton chain can be divided in two subchains:
   $\mathbf{{}^1H + p^+ \rightarrow {}^2D + e^+ + \nu_e}$, and then ${}^2\text{D} + \text{p} \rightarrow {}^3\text{He} + \gamma$.
   - I. pp1: ${}^3\text{He} + {}^3\text{He} \rightarrow 2\text{p}^+ + {}^4\text{He}$. There is 26,21 + (0,51) MeV released.
   - II. pp2: ${}^3\text{He} + \alpha \rightarrow {}^7\text{Be} + \gamma$
     - i. ${}^7\text{Be} + \text{e}^- \rightarrow {}^7\text{Li} + \nu$, dan ${}^7\text{Li} + \text{p}^+ \rightarrow 2\,{}^4\text{He} + \gamma$. 25,92 + (0,80) MeV.
     - ii. ${}^7\text{Be} + \text{p}^+ \rightarrow {}^8\text{B} + \gamma$, dan ${}^8\text{B} + \text{e}^+ \rightarrow 2\,{}^4\text{He} + \overline{\nu}$. 19,5 + (7,2) MeV.

     Both ^{7}Be chains become more important with raising T.
2. The CNO cycle. The first chain releases 25,03 + (1,69) MeV, the second 24,74 + (1,98) MeV. The reactions are shown below.

$$\begin{array}{ccccc}
 & & \longrightarrow & \searrow & \\
\nearrow & \rightarrow & {}^{15}\text{N} + \text{p}^+ \rightarrow \alpha + {}^{12}\text{C} & & {}^{15}\text{N} + \text{p}^+ \rightarrow {}^{16}\text{O} + \gamma \\
 & & \downarrow & & \downarrow \\
{}^{15}\text{O} + \text{e}^+ \rightarrow {}^{15}\text{N} + \overline{\nu} & & {}^{12}\text{C} + \text{p}^+ \rightarrow {}^{13}\text{N} + \gamma & & {}^{16}\text{O} + \text{p}^+ \rightarrow {}^{17}\text{F} + \gamma \\
\uparrow & & \downarrow & & \downarrow \\
\mathbf{{}^{14}N + p^+ \rightarrow {}^{15}O + \gamma} & & {}^{13}\text{N} \rightarrow {}^{13}\text{C} + \text{e}^+ + \nu & & {}^{17}\text{F} \rightarrow {}^{17}\text{O} + \text{e}^+ + \nu \\
 & & \downarrow & & \downarrow \\
\nwarrow & \leftarrow & {}^{13}\text{C} + \text{p}^+ \rightarrow {}^{14}\text{N} + \gamma & & {}^{17}\text{O} + \text{p}^+ \rightarrow \alpha + {}^{14}\text{N} \\
 & & \longleftarrow & \swarrow &
\end{array}$$