Note: By moments we mean the moments about origin or raw moments.

The first four moments about the origin are given by

1. $\mu_1' = E(X)$=Mean
2. $\mu_2' = E(X^2)$
3. $\mu_3' = E(X^3)$
4. $\mu_4' = E(X^4)$

Note: $Var(X) = E(X^2) - [E(X)]^2 = \mu_2' - {\mu_1'}^2$=Second moment - square of the first moment.

Definition 4.2 (Moments about mean or Central moments). *The r^{th} moment of a random variable X about the mean μ is defined as $E[(X-\mu)^r]$ and is denoted by μ_r.*

The first four moments about the mean are given by

1. $\mu_1 = E(X-\mu) = E(X) - E(\mu) = \mu - \mu = 0$
2. $\mu_2 = E[(X-\mu)^2] = Var(X)$
3. $\mu_3 = E[(X-\mu)^3]$
4. $\mu_4 = E[(X-\mu)^4]$

Definition 4.3 (Moments about any point a). *The r^{th} moment of a random variable X about any point a is defined as $E[(X-a)^r]$ and we denote it by m_r'.*

The first four moments about a point 'a' are given by

1. $m_1' = E(X-a) = E(X) - a = \mu - a$
2. $m_2' = E[(X-a)^2]$
3. $m_3' = E[(X-a)^3]$
4. $m_4' = E[(X-a)^4]$

Relation between moments about the mean and moments about any arbitrary point a

Let μ_r be the r^{th} moment about mean and m_r' be the r^{th} moment about any point a. Let μ be the mean of X.

Since the total sales X is in thousands of units, the sales between 500 and 1500 units is the event A which stands for $\frac{1}{2} = 0.5 < X < \frac{3}{2} = 1.5$ and the sales over 1000 units is the event B which stands for $X > 1$. $\implies \quad A \cap B = 1 < X < 1.5$

Now

$$\begin{aligned} P(A) &= P(0.5 < X < 1.5) = \int_{0.5}^{1.5} f(x)dx \\ &= \int_{0.5}^{1} x\ dx + \int_{1}^{1.5} (2-x)\ dx = \boxed{\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} P(B) &= P(X > 1) = \int_{1}^{2} f(x)dx \\ &= \int_{1}^{2} (2-x)\ dx = \boxed{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(1 < X < 1.5) = \int_{1}^{1.5} f(x)dx \\ &= \int_{1}^{1.5} (2-x)\ dx = \boxed{\frac{3}{8}} \end{aligned}$$

The condition for independent events: $P(A) \cdot P(B) = P(A \cap B)$

Here, $P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$

$\therefore \quad A$ and B are independent events.

Example: 11. *If a random variable X has the following probability distribution, find* $E(X),\ E(X^2),\ Var(X),\ E(2X+1),\ Var(2X+1)$.

x	-1	0	1	2
$p(x)$	0.3	0.1	0.4	0.2

Hints/Solution: Here X is a discrete RV. $\therefore$

$$\begin{aligned} E(X) &= \sum_{i=-\infty}^{\infty} x_i p(x_i) \\ &= (-1) \times 0.3 + 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.2 \\ &= -0.3 + 0 + 0.4 + 0.4 = \boxed{0.5} \end{aligned}$$

$$\text{Now } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$
$$= \int_{-\infty}^{0} x^2 \frac{1}{5} e^{-\frac{x}{5}} dx$$
$$= -\frac{1}{5}\left[(5x^2+50x+250)e^{-\frac{x}{5}}\right]_0^{\infty} = \boxed{50}$$

$$\therefore \quad Var(X) = 50 - [5]^2 = \boxed{25}$$

Example: 14. *Find the mean and standard deviation of the distribution*

$$f(x) = \begin{cases} kx(2-x), & \text{when } 0 <\leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Hints/Solution: Given that the continuous RV X whose pdf is given by

$$f(x) = \begin{cases} kx(2-x), & \text{when } 0 <\leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x)$ is a pdf,

$$\text{we have } \int_{-\infty}^{\infty} f(x)dx = 1$$
$$\text{i.e. } \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$
$$\text{i.e. } \int_{-\infty}^{0} 0\cdot dx + \int_{0}^{2} kx(2-x)\,dx + \int_{2}^{\infty} 0\cdot dx = 1$$
$$\text{i.e. } 0 + k\left[x^2 - \frac{x^3}{3}\right]_0^2 + 0 = 1$$
$$\Longrightarrow \boxed{k = \frac{3}{4}}$$

Hints/Solution: We know that

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1\\
\text{i.e.,}\ \int_{0}^{\infty} kx^2e^{-x}\,dx &= 1\\
\text{i.e.,}\ k\int_{0}^{\infty} e^{-x}x^{3-1}\,dx &= 1\\
\text{i.e.,}\ k\Gamma(3) &= 1 \quad \left[\because\ \Gamma(n)=\int_{0}^{\infty} e^{-x}x^{n-1}\,dx\right]\\
\text{i.e.,}\ k\cdot 2! &= 1 \implies k=\frac{1}{2}. \qquad [\because\ \Gamma(n)=(n-1)!]
\end{aligned}
$$

Now, The r^{th} moment is given by

$$
\begin{aligned}
\mu'_r &= E(X^r)\\
&= \int_{-\infty}^{\infty} x^r f(x)\,dx\\
&= \int_{0}^{\infty} x^r kx^2e^{-x}\,dx\\
&= k\int_{0}^{\infty} e^{-x}x^{r+3-1}\,dx\\
&= \frac{1}{2}\Gamma(r+3)=\frac{1}{2}(r+2)!
\end{aligned}
$$

$$
\begin{aligned}
\text{Now,}\quad \text{First Moment}\ \mu'_1 &= \frac{1}{2}(3)! = 3\\
\text{Second Moment}\ \mu'_2 &= \frac{1}{2}(4)! = 12\\
\text{Third Moment}\ \mu'_3 &= \frac{1}{2}(5)! = 60\\
\text{Fourth Moment}\ \mu'_4 &= \frac{1}{2}(6)! = 360
\end{aligned}
$$

Example: 20. *A random variable X has the pdf $f(x)=\frac{1}{2}e^{-\frac{x}{2}},\ \ x\geq 0$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Hints/Solution: The MGF of X is given by

$$
\begin{aligned}
\text{Given } \mu'_r &= (r+1)!2^r \\
\therefore\ \mu'_1 &= 2!2 \\
\mu'_2 &= 3!2^2 \\
\mu'_3 &= 4!2^3 \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ M_X(t) = E(e^{tX}) &= 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \cdots + \frac{t^r}{r!}\mu'_r + \ldots \\
&= 1 + \frac{t}{1!}2 + \frac{t^2}{2!}3!2^2 + \frac{t^3}{3!}4!2^3 + \ldots \\
&= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \ldots \\
\therefore\ M_X(t) &= (1-2t)^{-2}.
\end{aligned}
$$

Now, Differentiating w.r.to t, we get

$$
\begin{aligned}
M'_X(t) &= -2(1-2t)^{-3}(-2) \\
M''_X(t) &= 6(1-2t)^{-4}(-2)^2 \\
\text{Now,}\quad \text{First Moment=Mean}= E(X) = \mu'_1 &= M'_X(0) = 4 \\
\text{Second Moment}= E(X^2) = \mu'_2 &= M''_X(0) = 24 \\
\therefore\ Var(X) &= E(X^2) - [E(X)]^2 = 24 - 16 = 8.
\end{aligned}
$$

Find the (i) the value of A,(ii) the Distribution Function (CDF) (iii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$. (iv) the probability that in a day the sales is (a) more than 500 kgs (b) less than 500 kgs (c) between 250 and 750 kgs. Ans: $A = \frac{1}{25}$

9. The cumulative distribution function (CDF) of a random variable X is given by

$$F(x) = \begin{cases} 1 - \frac{4}{x^2}, & \text{when } x > 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the pdf of X,(ii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$ (iii) $P(X < 3)$, $P(3 < X < 5)$.

10. A coin is tossed until a head appears. What is the expected value of the number of tosses?. Also find its variance.

11. The pdf of a random variable X is given by

$$f(x) = \begin{cases} a + bx, & \text{when } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the value of a, b if the mean is 1/2,(ii) the variance of X (iii) $P(X > 0.5/X < 0.5)$

12. The first three moments about the origin are 5,26,78. Find the first three moments about the value x=3. Ans: 2,5,-48

13. The first two moments about x=3 are 1 and 8. Find the mean and variance. Ans: 4,7

14. The pdf of a random variable X is given by

$$f(x) = \begin{cases} k(1-x), & \text{when } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the the value of k,(ii) the r^{th} moment about origin (iii) mean and variance. Ans: $k = 2$

15. An unbiased coin is tossed three times. If X denotes the number of heads appear, find the MGF of X and hence find the mean and variance.

16. Find the MGF of the distribution whose pdf is $f(x) = ke^{-x}$, $x > 0$ and hence find its mean and variance.

17. The pdf of a random variable X is given by

$$f(x) = \begin{cases} x, & \text{when } 0 \le x \le 1 \\ 2 - x, & \text{when } 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

For this find the MGF and prove that mean and variance cannot be find using this MGF and then find its mean and variance using expectation.