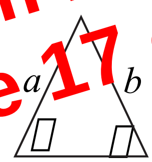
> Now the angles α and γ form a straight line and so are supplementary, it follows that $\alpha = 180° - \gamma$. Similarly, β and γ also form a straight line and so again we have that $\beta = 180° - \gamma$. So we have $\alpha = 180° - \gamma = \beta$, which shows that the angles are congruent.

One last note on notation. Throughout the book we will tend to use capital letters $(A, B, C, \ldots)$ to represent points and lowercase letters $(a, b, c, \ldots)$ to represent line segments or length of line segments. While it a goal to be consistent it is not always convenient, however it should be clear from the context what we are referring to whenever the notation varies.

2.4 Isoceles triangles

A special group of triangles are the isoceles triangles. The root *iso* means "same" and isoceles triangles are triangles that have at least two sides of equal length. A useful fact from geometry is that if two sides of the triangle have equal length then the corresponding angles (i.e. the angles *opposite* the sides) are congruent. In the picture below it means that if $a = b$ then $\alpha = \beta$.



The geometrical proof goes like this. Pick up and "turn over" the triangle and put it back down on top of the old triangle keeping the vertex where the two sides of equal length come together at the same point. The triangle that is turned over will exactly match the original triangle and so in particular the angles (which have now traded places) must also exactly match, i.e., they are congruent.

A similar process will show that if two angles in a triangle are congruent then the sides opposite the two angles have the same length. Combining these two fact means that in a triangle having equal sides is the same as having equal angles.

One special type of isoceles triangle is the *equilateral* triangle which has all three of the sides of equal length. Applying the above argument twice shows that all the angles of such a triangle are congruent.

2.5 Right triangles

In studying triangles the most important triangles will be the *right triangles*. Right triangles, as the name implies, are triangles with a right angle. Triangles can be

> *Solution* In this triangle we are given the lengths of the "legs" (i.e. the sides joining the right angle) and we are missing the hypotenuse, or c. And so in particular we have that
>
> $$3^2 + 7^2 = c^2 \quad \text{or} \quad c^2 = 58 \quad \text{or} \quad c = \sqrt{58} \approx 7.616$$

Note in the example that there are two values given for the missing side. The value $\sqrt{58}$ is the *exact* value for the missing side. In other words it is an expression that refers to the unique number satisfying the relationship. The other number, 7.616, is an approximation to the answer (the '$\approx$' sign is used to indicate an approximation). Calculators are wonderful at finding approximations but bad at finding exact values. Make sure when answering the questions that your answer is in the requested form.

Also, when dealing with expressions that involve square roots there is a temptation to simplify along the following lines, $\sqrt{a^2 + b^2} = a + b$. This seems reasonable, just taking the square root of each term, but it is not correct. Erase any thought of doing this from your mind.

This does not work because there are several operations going on in this relationship. There are terms being squared, terms being added and terms having the square root taken. Rules of algebra dictate which operations must be done first, for example one rule says that if you are taking a square root of terms being added together you first must add then take the square root. Most of the rules of algebra are intuitive and so do not worry too much about memorizing them.

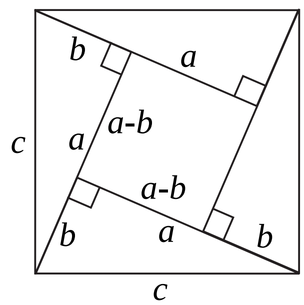
3.2 The Pythagorean theorem and dissection

There are literally hundreds of proofs for the Pythagorean theorem. We will not try to go through them all but there are books that contain collections of proofs of the Pythagorean theorem.

Our first method of proof will be based on the principle of *dissection*. In dissection we calculate a value in two different ways. Since the value doesn't change based on the way that we calculate it, the two values that are produced will be equal. These two calculations being equal will give birth to relationships, which if done correctly will be what we are after.

For our proof by dissection we first need something to calculate. So starting with a right triangle we will make four copies and place them as shown below. The result will be a large square formed of four triangles and a small square (you should verify that the resultant shape is a square before proceeding).

The value that we will calculate is the area of the figure. First we can compute the area in terms of the large square. Since the large square has sides of length c the area of the large square is c^2.



The second way we will calculate area is in terms of the pieces making up the large square. The small square has sides of length $(a-b)$ and so its area is $(a-b)^2$. Each of the triangles has area $(1/2)ab$ and there are four of them.

Putting all of this together we get the following.

$$c^2 = (a-b)^2 + 4 \cdot \frac{1}{2}ab = (a^2 - 2ab + b^2) + 2ab = a^2 + b^2$$

3.3 Scaling

Imagine that you made a sketch on paper made out of rubber and then stretched or squished the paper in a nice uniform manner. The sketch that you made would get larger or smaller, but would always appear essentially the same.

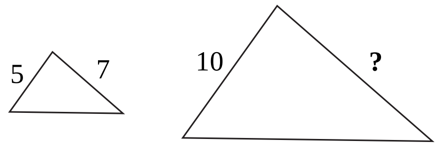
This process of stretching or shrinking is *scaling*. Mathematically, scaling is when you multiply all distances by a positive number, say k. When $k > 1$ then we are stretching distances and everything is getting larger. When $k < 1$ then we are shrinking distances and everything is getting smaller.

What effect does scaling have on the size of objects?

Lengths: The effect of scaling on paths is to multiply the total length by a factor of k. This is easily seen when the path is a straight line, but it is also true for paths that are not straight since all paths can be approximated by straight line segments.

Areas: The effect of scaling on areas is to multiply the total area by a factor of k^2. This is easily seen for rectangles and any other shape can be approximated by rectangles.

Volumes: The effect of scaling on volumes is to multiply the total volume by a factor of k^3. This is easily seen for cubes and any other shape can be approximated by cubes.



only need to multiply the length of 7 by our scaling factor to get our final answer.

To figure out the scaling factor, we note that the side of length 5 became a side of length 10. In order to achieve this we had to scale by a factor of 2. So in particular, the length of the indicated side is 14.

3.4 The Pythagorean theorem and scaling

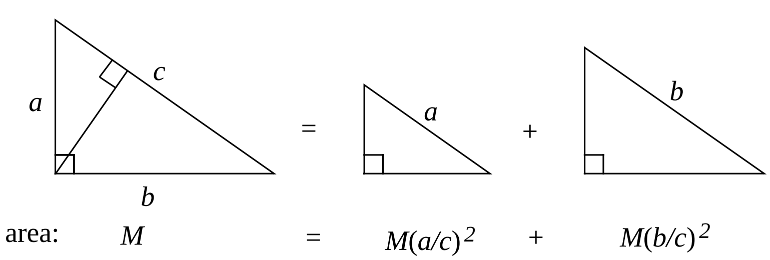
To use scaling to prove the Pythagorean theorem we must first produce some similar triangles. This is done by cutting our right triangle up into two smaller right triangles, which are similar as shown below. So in essence we now have three right triangles all similar to one another, or in other words they are scaled versions of each other. Further, these triangles will have hypotenuses of length a, b and c.

To get from a hypotenuse of length c to a hypotenuse of length a we would scale by a factor of (a/c). Similarly, to get from a hypotenuse of length c to a hypotenuse of length b we would scale by a factor of (b/c).

In particular, if the triangle with the hypotenuse of c has area M then the triangle with the hypotenuse of a will have area $M(a/c)^2$. This is because of the effect that scaling has on areas. Similarly, the triangle with a hypotenuse of b will have area $M(b/c)^2$.

But these two smaller triangles exactly make up the large triangle. In particular, the area of the large triangle can be found by adding the areas of the two smaller triangles. So we have,

$$M = M\left(\frac{a}{c}\right)^2 + M\left(\frac{b}{c}\right)^2 \qquad \text{which simplifies to} \qquad c^2 = a^2 + b^2.$$



Lecture 4

Angle measurement

In this lecture we will look at the two popular systems of angle measurement, degrees and radians.

4.1 The wonderful world of π

The number π (pronounced like "pie") is among the most important numbers in mathematics. It arises in a wide array of mathematical applications, such as statistics, mechanics, probability, and so forth. Mathematically, π is defined as follows:

$$\pi = \frac{\text{circumference of a circle}}{\text{diameter of a circle}} \approx 3.14159265\ldots$$

Since any two circles are scaled versions of each other it does not matter what circle is used to find an estimate for π.

> **Example 1** Use the following scripture from the King James Version of the Bible to estimate π.
>
> And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about. – 1 Kings 7:23
>
> *Solution* The verse describes a round font with a diameter of approximately 10 cubits and a circumference of approximately 30 cubits. Using the definition of π we get.
>
> $$\pi \approx \frac{30}{10} = 3$$

Note that this verse does not give an exact value of π, but this should not be too surprising and is most likely attributed to a rounding or measurement error.

4.2 Circumference and area of a circle

From the definition of π we can solve for the circumference of a circle. From which we get the following,

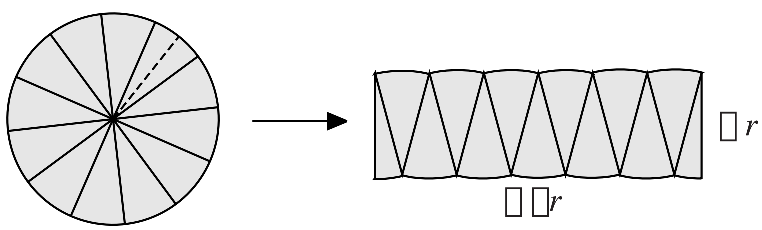
$$\begin{aligned} \text{circumference} &= \pi \cdot (\text{diameter}) \\ &= 2\pi r \qquad \text{(where } r \text{ is the radius of the circle).} \end{aligned}$$

The diameter of a circle is how wide the circle is at its widest point. The radius of the circle is the distance from the center of the circle to the edge. Thus the diameter which is all the way across is twice the radius which is half-way across.

One of the great observations of the Greeks was connecting the number π which came from the circumference of the circle to the area of the circle. The idea connecting them runs along the following lines. Take a circle and slice it into a large number of pie shaped wedges. Then take these pie shaped wedges and rearrange them to form a shape that looks like a rectangle with dimensions of half the circumference and the radius. As the number of pie shaped wedges increases the shape looks more and more like the rectangle and so the circle has the same area as the rectangle, so we have,

$$\text{area} = \left(\frac{1}{2}(2\pi r)\right) r = \pi r^2.$$

Pictorially, this is seen below.



4.3 Gradians and degrees

The way we measure angles is somewhat arbitrary and today there are two major systems of angle measurement, degrees and radians, and one minor system of angle measurement, gradians.

Gradians are similar to degrees but instead of splitting up a circle into 360 parts we break it up into 400 parts. Gradians are not very widely used and this will be our only mention of them. Even though it is not a widely used system to find our angle that we want we can subtract off 18 revolutions and the result will be an angle between 0° and 360°. So our final answer is,

$$6739° - 18 \cdot 360° = 259°.$$

4.4 Minutes and seconds

It took mathematics a long time to adopt our current decimal system. For thousands of years the best way to represent a fraction of a number was with fractions (and sometimes curiously so). But they needed to be able to measure just a fraction of an angle. To accommodate this they adopted the system of minutes and seconds.

One minute (denoted by $'$) corresponds to 1/60 of a degree. One second (denoted by $''$) correspond to 1/60 of a minute, or 1/3600 of a degree. This is analogous to our system of time measurement where we think of a degree representing one hour.

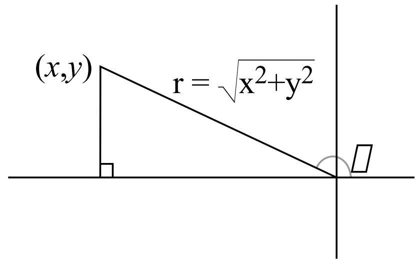
This system of degrees and minutes allowed for accurate measurement. For example, $1''$ is to 360° as 1 second is to 15 days. As another example, if we let the equator of the earth correspond to 360° then one second would correspond to about 101 feet.

Most commonly the system of minutes and seconds is used today in cartography, or map making. For example, Mount Everest is located at approximately $27°59'16''$ north latitude and $86°55'40''$ west longitude.

The system of minutes and seconds is also used in other places such as com woodworking machines, but for the most part it is not commonly used. In addition most handheld scientific calculators are also equipped to convert between the decimal system and $D°M'S''$. For these two reasons we will not spend time mastering this system.

Example 3 Convert 51.1265° to $D°M'S''$ form.

Solution It's easy to see that we will have 51°, it is the minutes and seconds that will pose the greatest challenge to us. Since there are $60'$ in one degree, to convert .1265° into minutes we multiply by 60. So we get that $.1265° = 7.59'$. So we have $7'$. Now we have $.59'$ to convert to seconds. Since there are $60''$ in one minute, to convert $.59'$ into seconds we multiply by 60. So we get that $.59' = 35.4''$. Combining this altogether we have $51.1265° = 51°7'35.4''$.



Here $r = \sqrt{x^2 + y^2}$ is the distance to the origin and will always be positive. Then we can define the trigonometric functions in terms of x, y and r as follows,

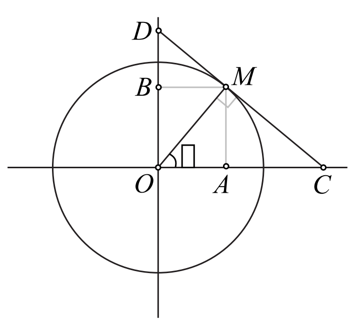
$$\begin{array}{lll} \sin(\theta) = \dfrac{y}{r}, & \cos(\theta) = \dfrac{x}{r}, & \tan(\theta) = \dfrac{y}{x}, \\ \csc(\theta) = \dfrac{r}{y}, & \sec(\theta) = \dfrac{r}{x}, & \cot(\theta) = \dfrac{x}{y}. \end{array}$$

This works by taking any point in the plane (x, y) and associating it with (or scaling it to) a point on the unit circle, namely the point $(x/r, y/r)$, which is associated with the same angle θ. We then let the trigonometric functions for the point (x, y) to be defined as the trigonometric functions for the point on the unit circle $(x/r, y/r)$.

In particular, as with the unit circle, we can associate every point in the plane except the origin with an angle. The idea of taking a point and scaling it to a point on the unit circle will play an important role later on.

6.9 Supplemental problems

1. Given that the circle shown below is the unit circle match each of the six trigonometric functions for the angle θ to one of the following lengths, OA, OB, OC, OD, MC and MD. *Hint*: find the angle formed by going from O to D to C in terms of θ.



When we deal with graphing the trigonometric functions we will *always* work in radians. This is not because we cannot graph in degrees, but rather there are some deeper hidden reasons which come from calculus as to why we choose radians. We will catch a glimpse of these reasons later on.

7.3 Over and over and over again

If we were to graph the sine and cosine curves correctly we would have to put in a lot of values. However, we can save ourselves some work by making an observation. We know that if two angles are co-terminal they will have the same values for the trigonometric functions, for example $\sin(x + 2\pi) = \sin(x)$. In particular the trigonometric functions are repeating.

To make a graph of the trigonometric function we only need to determine what it looks like on an interval that contains a complete revolution. Once we have that we just copy it over and over to get the complete graph for the function.

Functions that have this property are called periodic and the minimum amount of time it takes to repeat is the period. The sine and cosine functions are 2π periodic while the tangent function is π periodic.

7.4 Even and odd functions

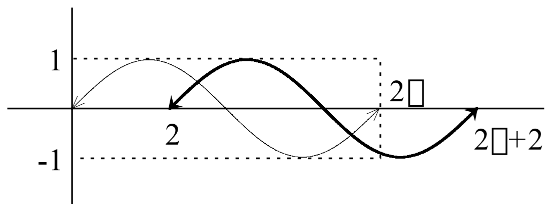
The graphs of some functions exhibit symmetry. There are two special types of symmetry that we will encounter when graphing functions.

The first type of symmetry is around the y axis. Imagine graphing the function then folding it in half along the y axis. If the two halves exactly match up then it is symmetrical around the y axis. Such a function is called an *even* function and satisfies the relationship $f(-x) = f(x)$. Examples of even trigonometric functions are the cosine and secant functions.

The second type of symmetry is around the origin. Imagine graphing the function then rotating the graph a half revolution around the origin. If it looks the same as before then it is symmetrical around the origin. Such a function is called an *odd* function and satisfies the relationship $f(-x) = -f(x)$. Examples of odd trigonometric functions are the sine, cosecant, tangent and cotangent functions.

Example 1 Determine whether the following function is even, odd or neither.

$$f(x) = \sin^2(x) - \cos(x)$$



7.6 The wild and crazy inside terms

In graphing functions changing the inside terms seems to do things that are counter-intuitive. As an example consider the function $y = a\sin(bx - c) + d$. This function will have period $2\pi/b$ and a horizontal shift of c/b. Not what we would expect.

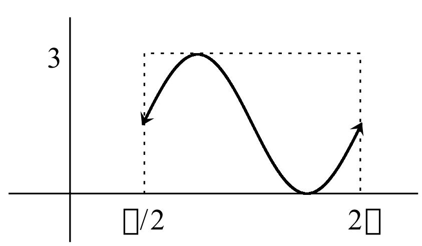
To see where these strange values arise recall that one period of the sine curve corresponds to one revolution around the circle. So one period begins at 0 and ends at 2π. If we are interested in exploring one period of our modified curve we would do it by finding when the inside expression is 0 (this is the start of the period) and when it is 2π (this is the end of the period). In particular we have the following,

start	$bx - c = 0$	or	$x = (c/b)$
end	$bx - c = 2\pi$	or	$x = (2\pi/b) + (c/b)$.

Note that the start of the period is now at the value c/b, this is why our horizontal shift is c/b. The difference between the start and the end represents the period, that is how long it takes to repeat, and so the period will be $2\pi/b$.

Example 2 Given that the graph shown below is one period of the sine curve find the amplitude, vertical shift, period and horizontal shift. Using these values write an equation for the curve in the form,

$$y = a\sin(bx - c) + d$$



Solution From the graph the height between the lowest and highest values is 3, and so the amplitude is half that or $3/2$. The vertical shift (keeping in mind that the sine curve should start at the origin) is $3/2$. The period is the length from the beginning to the end and so is $2\pi - \pi/2 = 3\pi/2$. Finally, the horizontal shift is $\pi/2$.

With these values in hand we can now start finding a, b c and d. The amplitude is a and so $a = 3/2$. The vertical shift is d and so $d = 3/2$. The period is $2\pi/b$ so $2\pi/b = 3\pi/2$ or $b = 4/3$. The horizontal shift is c/b so $(c = b\pi/2 = 2\pi/3)$. Putting these together we have,

$$y = \frac{3}{2}\sin\left(\frac{4}{3}x - \frac{2\pi}{3}\right) + \frac{3}{2}.$$

In this example we had a specific period of the sine curve given to us. What if we were given the whole sine curve and were asked to find an expression of the form $y = a\sin(bx - c) + d$, which period should we use? The correct answer is any of them. You can choose any full period to determine your constants. Note that the constants will depend upon which period you choose but they will all correspond to the same curve.

Example 2 Given the following function find the amplitude, vertical shift, period and horizontal shift. Then use these values to graph one period of the function.

$$y = 2\sin\left(\pi x + \frac{\pi}{3}\right) - 3$$

Answer: From the equation we can read off the amplitude, which is $a = 2$ and the vertical shift which is $d = -3$. To find the period we take the value $b = \pi$ and divide it into 2π which gives a period of $(2\pi/\pi) = 2$. To find the horizontal shift we take the value of $c = -\pi/3$ and divide it by $b = \pi$ to get a horizontal shift of $(-\pi/3)/\pi = -1/3$.

To graph the function we first use the vertical and horizontal shift to find where the curve starts. We can then use the information about the amplitude and the period to draw a box that will tightly contain one period of the curve. The box for our problem is shown below on the left. With the box in place we then draw in one period of the sine curve, exactly filling the box, to get our required graph. This is shown below on the right.

Lecture 9

Working with trigonometric identities

In this lecture we will expand upon our trigonometric skills by learning how to manipulate and verify trigonometric identities.

9.1 What the equal sign means

In mathematics we often will use the '=' sign with two different meanings in mind. Namely, it is used to denote identities and conditional relationships.

An *identity* represents a relationship that is *always* true. We have seen several examples of this. For instance the Pythagorean identity, $\cos^2(\theta) + \sin^2(\theta) = 1$ is true for every value of θ and so is an identity.

A *conditional relationship* represents an equation that is *sometimes (possibly never)* true. We have also seen examples of this. For instance in the last lecture we found that the relationship $\cos(\theta) = 2/3$ is satisfied for some but not all θ.

So the '=' sign gets a lot of usage and you need to be careful to see whether it is being used to represent an identity or a conditional relationship. (Some mathematical zealots will use the '$\equiv$' sign to denote an identity, we shall not adopt this practice here.)

For now we will focus on identities and save looking at conditional relationships for later. The most important part of working with identities is being able to manipulate them, bend them to your will so to speak. To learn how to do this we will look at a variety of techniques from algebra.

9.3 The conju-what? The conjugate

One very useful algebraic trick to use in simplifying some expressions is the *conjugate*. The conjugate basically means change the sign in the middle. So for example the conjugate of $1+\cos(\theta)$ is $1-\cos(\theta)$ (i.e. we changed the sign in the middle). This is useful because when multiplying conjugates the "cross terms" cancel, that is,

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2.$$

Use of the conjugate is particularly helpful in getting terms that have expressions like $1 \pm \cos(x)$ or $1 \pm \sin(x)$ in the denominator out of fractional form. This is because of the Pythagorean identities. For example,

$$(1-\cos(x))(1+\cos(x)) = 1 - \cos^2(x) = \sin^2(x).$$

Example 2 Rewrite the following expression so that it is not in fractional form.

$$\frac{1}{1+\sin(x)}$$

Solution. We will start with the expression and multiply through both the top and the bottom by the conjugate. (We need to multiply both the top and the bottom so that the total value of the expression does not change.) Doing this, we get the following.

$$\begin{aligned}
\frac{1}{1+\sin(x)} &= \frac{1(1-\sin(x))}{(1+\sin(x))(1-\sin(x))} = \frac{1-\sin(x)}{1-\sin^2(x)} \\
&= \frac{1-\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos(x)}\frac{1}{\cos(x)} \\
&= \sec^2(x) - \tan(x)\sec(x)
\end{aligned}$$

Note that in this example we broke up the fraction into two pieces. This is no problem when we break up addition in the numerator, i.e. $(a+b)/c = (a/c)+(b/c)$, but does not work for terms in the denominator.

9.4 Dealing with square roots

Sometimes when dealing with expressions we will need to work with square roots. When doing so there are some important things to remember. The first is that the square root does not break up over addition, i.e. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$, but does break up over multiplication, i.e. $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

The second is that the expression $\sqrt{x^2}$ does not always equal x, but rather $\sqrt{x^2} = |x|$. In other words, if you square a number and then take the square root you will be left with the absolute value of what you started with. You can drop the absolute value sign when you are certain that the value will be nonnegative.

Example 3 Simplify so as to remove the square root in the following expression.

$$\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

Solution Note that in the denominator inside the square root that we have $1+\cos(\theta)$. This is a wonderful expression to use conjugates with. So starting by multiplying through by the conjugates, we will get the following:

$$\begin{aligned}\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} &= \sqrt{\frac{(1-\cos(\theta))(1-\cos(\theta))}{(1+\cos(\theta))(1-\cos(\theta))}} = \sqrt{\frac{(1-\cos(\theta))^2}{1-\cos^2(\theta)}} \\ &= \sqrt{\frac{(1-\cos(\theta))^2}{\sin^2(\theta)}} = \frac{|1-\cos(\theta)|}{|\sin(\theta)|} = \frac{1-\cos(\theta)}{|\sin(\theta)|}\end{aligned}$$

In the last step we can drop the absolute value sign on the term $1-\cos(\theta)$ because it will *always* be nonnegative, or in other words bigger than or equal to zero. But we cannot drop the absolute value sign on the term $\sin(\theta)$ because it can sometimes be negative.

9.5 Verifying trigonometric identities

Up to this point we have not been verifying identities but just putting tools in place to simplify expressions. Verifying an identity requires simplifying one expression to another expression.

When verifying identities the following guidelines are helpful to keep in mind.

thing to do, but there is a subtle reason that we cannot. When solving conditional relationships, we are looking at x over all possibilities and trying to determine which ones satisfy the relationship. When we divide both sides of the equation by $\cos(x)$ then when $\cos(x)$ is 0 we are dividing by 0 which is bad mathematics. So we have the following rule in solving conditional relationships: *You cannot divide to cancel terms if the term is ever zero.*

10.3 Use the identities

Sometimes grouping similar terms together and factoring will not be enough, so we start getting more sophisticated. We will sometimes need to turn to the identities to help us. The identities can be used in several ways, primary is their ability to simplify complex expressions that might be on one side of the equation. In particular they can be used in combining terms.

Example 3 Solve for all the angles for which the following conditional relationship is satisfied.

$$\sin(x) = \cos(x)$$

Solution In this example we cannot combine the terms since they are not similar and if we grouped the terms on one side, we would not be able to factor out any common expression. So after staring at the equation for some time we come up with a plan. Namely, all we have here is the sine and cosine function, and the tangent function is the sine over the cosine. So let us divide both sides by the term $\cos(x)$ (here it will be alright to divide because we are not cancelling terms). So we have the following,

$$\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)} \qquad \text{or} \qquad \tan(x) = 1.$$

Now we have gotten to our ideal situation, a function being equal to a number. Looking up on our chart we see that the tangent function is 1 when our angle is 45° (or $\pi/4$). The tangent function is nice because it is periodic with period of 180° or π and so our final solution is,

$$45^\circ + k180^\circ \text{ for } k = 0, \pm1, \pm2, \ldots \text{ or}$$
$$\pi/4 + k\pi \text{ for } k = 0, \pm1, \pm2, \ldots.$$

Lecture 11

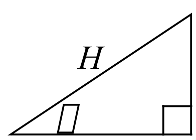
The sum and difference formulas

In this lecture we will learn how to work with terms such as $\sin(x + y)$. Along the way we will learn the useful tool of projection.

11.1 Projection

A proper discussion of projection must wait until later. For now we will use a very simple and straightforward version. Namely, given a hypotenuse of a right triangle and an acute angle we will find expressions for the lengths of the legs of the right triangle.

To find our formulas for projection consider the picture below where we know the length of the hypotenuse (which we will call H) and the acute angle θ.

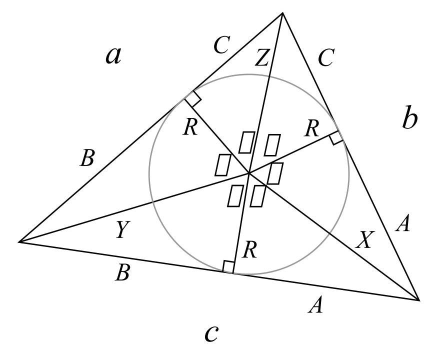


Using the definition of trigonometric functions as ratios of right triangles we can find the length of the missing sides. So we have,

$$\sin(\theta) = \frac{opp}{H} \qquad \text{or} \qquad opp = H\sin(\theta),$$
$$\cos(\theta) = \frac{adj}{H} \qquad \text{or} \qquad adj = H\cos(\theta).$$

So knowing the length of the hypotenuse and an acute angle we can then fill in the lengths of the other sides of the triangle, as is shown below.

To see why we use the name projection, imagine standing directly over the triangle with a bright flashlight. If we point our flashlight straight down the where the side and the circle just touch this will form a right angle and so we will have right triangles. In fact, we will have a total of six right triangles. To help us work with these triangles we label everything we can. Doing this we get the picture below.



12.4 The little ones

With our picture in place it is quite quick to add up the area of all these little triangles. Since they are all right triangles we can use some simple formulas and we will get that the total area is,

$$\text{area} = 2\cdot\frac{1}{2}AR + 2\cdot\frac{1}{2}BR + 2\cdot\frac{1}{2}CR = (A+B+C)R.$$

We would be done except that we do not know what A, B, C and R are. What we do know are the lengths of the sides of the big triangle, i.e., a, b and c. We need to find a way to rewrite A, B, C and R as expressions of a, b and c.

12.5 Rewriting our terms

From the triangle we get the relationships

$$A + C = b, \qquad A + B = c, \qquad B + C = a.$$

Now if this were an algebra book we would take some time to take these three equations and solve for A, B and C in terms of a, b and c. But we are here to learn about trigonometry and so we will jump to the end and get the following,

$$A = \frac{1}{2}(-a+b+c), \qquad B = \frac{1}{2}(a-b+c), \qquad C = \frac{1}{2}(a+b-c).$$

apply the double angle formulas from above and get,

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25},\\ \cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}.\end{aligned}$$

An amazing thing about these identities is how much information we can get without actually knowing what the angle θ is.

Starting with the double angle identity for the cosine function we can use the Pythagorean identity to rewrite it in different ways. Namely, we can have the following,

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= \cos^2(x)-(1-\cos^2(x))\\ &= 2\cos^2(x)-1,\end{aligned}$$

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= (1-\sin^2(x))-\sin^2(x)\\ &= 1-2\sin^2(x).\end{aligned}$$

13.2 Power reduction identities

Starting with these last two forms for $\cos(2x)$ we can manipulate and solve for the terms $\cos^2(x)$ and $\sin^2(x)$.

$$\cos(2x) = 2\cos^2(x)-1 \qquad \text{so} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\cos(2x) = 1-2\sin^2(x) \qquad \text{so} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

These are called the power reduction identities since we start with the term on the left hand side with a square power and the terms on the right side do not have square power terms in them (i.e. we reduced the highest power term by one).

We can use these formulas multiple times (sometimes in conjunction with other identities) to reduce expressions with powers higher than degree two.

Example 2 Rewrite $\sin^4(x)$ to an expression that does not have any terms with a power greater then one or two different trigonometric functions multiplied together.

Lecture 15

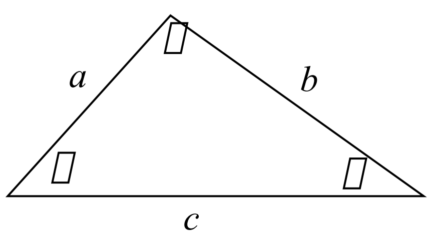
Law of sines and cosines

In this lecture we will introduce the law of sines and cosines which will allow us to explore oblique triangles.

15.1 Our day of liberty

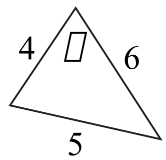
We can now free ourselves from using only right triangles and be able to work with all sorts of triangles. We will do it by introducing the law of sines and the law of cosines. Our derivation of these laws will be through use of right triangles, but these laws will let us put the right triangles in the background once proved.

For our notation in this lecture we will let a, b and c represent the length of the sides of a triangle while the quantities α, β and γ will represent the measure of the corresponding angles. Namely, they will match up according to the picture below.



15.2 The law of sines

For this law, start with any arbitrary triangle and from one of the vertices draw a line straight down to the base. This will split the triangle up into two smaller right triangles, such as is shown below,



and one angle. In order to use the law of cosines we need to know at least three of the four. Since we know the length of all of the sides we are okay to proceed with using the law of cosines. Doing so we get the following,

$$\cos(\theta) = \frac{4^2+6^2-5^2}{2(4)(6)} = \frac{9}{16} \quad \text{so} \quad \theta = \arccos\left(\frac{9}{16}\right) \approx 55.77°.$$

15.4 The triangle inequality

From the law of cosines we can derive a very important mathematical rule. First, recall that the cosine function has its range of values between -1 and 1 and in particular $-\cos(\gamma) \leq 1$. With this in mind, consider the following.

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \leq a^2 + 2ab + b^2 = (a+b)^2$$

By taking the square roots of both sides we get the following.

Triangle inequality $c \leq a + b$

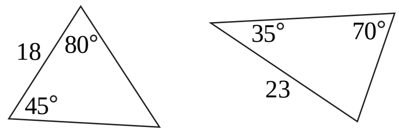
This also has the alternate forms $a \leq b + c$ and $b \leq a + c$.

In words, the triangle inequality says the following, *the direct route is the shortest.* If you want to move from one point on a triangle to another then going on the segment that connects the two points will always have you travel a distance that is less than or equal to going along the other two segments.

One of the most useful properties of the triangle inequality is to test whether or not you have a triangle. If you add up the two shortest sides of a triangle and it is less than the longest side, then it is no triangle at all.

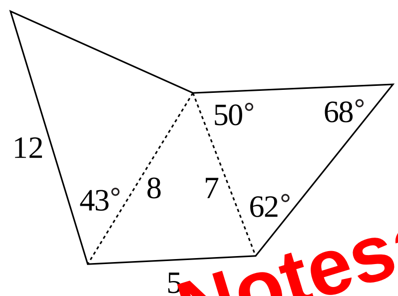
Notice that in the triangle inequality we have "less than or equal to," what would happen if we had equality? This would form a strange looking "triangle," namely, the triangle would not look like a triangle but rather a line segment. Sometimes there is concern over whether this truly is a triangle. At any rate it is good to think of it as an "extreme" example of a triangle. (Often times by studying extreme examples, i.e., worst case scenarios, we can get an idea of some behavior of an object.)

The triangle inequality is used extensively in mathematics. Particularly in calculus and any branch of mathematics that has to deal with measurement of space.



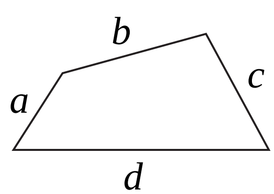
5. Using the formula from the previous question find the area of the triangles shown below. Round your answers to two decimal places.

6. Using only the information shown in the picture below find the total area. Round your answer to two decimal places.



Hint: this has been broken up into three triangles and we have three formulas to find area that use information about the length and the angles.

7. Using the triangle inequality show that $d \leq a + b + c$ in the figure below. *Hint*: to use the triangle inequality you will need a triangle.

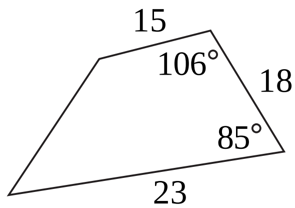


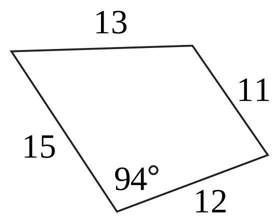
8. True/False. Since by the triangle inequality we have that $c \leq a + b$ then it is impossible for $c^2 > a^2 + b^2$.

9. The law of sines and cosines are well known, but there is also a lesser known law, called the law of tangents.

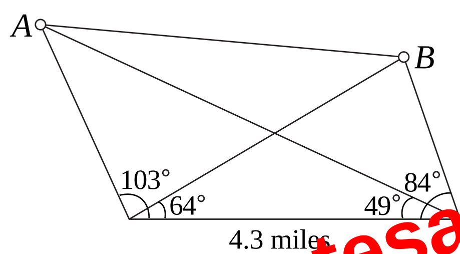
$$\textbf{Law of tangents} \qquad \frac{\tan(\alpha + \beta/2)}{\tan(\alpha - \beta/2)} = \frac{a+b}{a-b}$$

Verify the law of tangents formula.



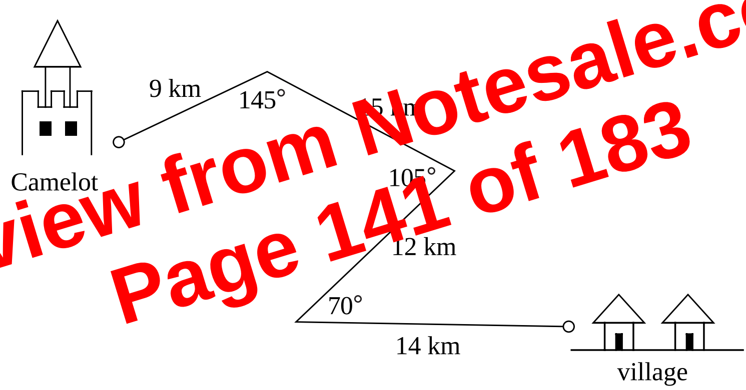


are a known distance apart and get the picture shown below. Using the given information find the distance between A and B (something which is almost impossible to measure directly). Round your answer to one decimal place.



8. King Arthur has recently decided to find the speed of the various birds that he has encountered. In particular he has recently acquired a pet parrot named Polly for which he wants to determine Polly's airspeed velocity. To do this he has Polly trained to always fly back to Camelot and then one day sends Polly out to a nearby village and at noon released, at the same time ye olde royal hour glass in Camelot starts keeping track of time. Polly makes the trip from the village to the castle in 93 minutes. However, King Arthur forgot to get an estimate on the distance as a parrot flies between the village and Camelot. He turns to you as the royal trigonometrist to determine the straight line distance from the village to Camelot and gives you the information shown below.

   Determine the straight line distance and then use this information to determine the airspeed velocity of Polly measured in kilometers per hour. Round your answers to the nearest whole number.


Camelot
9 km
145°
15 km
105°
12 km
70°
14 km
village

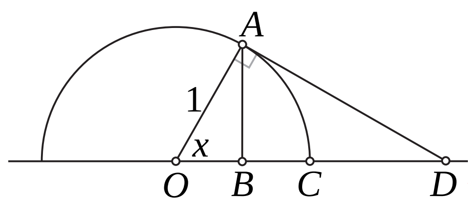
18.4 A limit involving trigonometry

With the idea of the squeezing principle in hand we can now evaluate an important limit. Specifically we are going to determine what happens to the values of the function,

$$\frac{\sin(x)}{x},$$

as the value of x (measured in radians) approaches 0.

Consider the diagram shown below.



From the diagram we have that $\overline{AB} \le \overline{AC} \le \overline{AD}$. Further, we can find the values of these lengths in terms of x. To find $\overline{AB}$ we can use the right triangle at the points A, B and the origin which has a hypotenuse of length 1 and an acute angle x to get that $\overline{AB} = \sin(x)$. Similarly we can get $\overline{AD} = \tan(x)$. To find the length $\overline{AC}$ we note that it is an arc of a circle with radius 1 and central angle x (with x measured in radians), so from geometry we have that $\overline{AC} = x$.

It follows that for x between 0 and $\pi/2$ radians that,

$$\sin(x) \le x \le \tan(x).$$

In a fraction if you make a denominator smaller then the total value gets larger and if you make the denominator larger the total value gets smaller. In particular using the relationship we just found we have,

$$\cos(x) = \frac{\sin(x)}{\tan(x)} \le \frac{\sin(x)}{x} \le \frac{\sin(x)}{\sin(x)} = 1.$$

We have now been able to put the function $\sin(x)/x$ in between the two functions 1 and $\cos(x)$ both of which go to 1 as x gets closer and closer to 0. Therefore we can apply the squeezing principle and conclude that the function $\sin(x)/x$ will also go to 1 as x goes to 0. Using mathematical notation we would say this in the following way,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Example 1 Using the relationship,

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1,$$

for x between 0 and $\pi/2$ radians show that,

$$\frac{\sin(2x)}{2} \leq \frac{\sin^2(x)}{x} \leq \sin(x),$$

is also satisfied for x between 0 and $\pi/2$ radians. From this, find what happens to the values of $\sin^2(x)/x$ as x approaches 0.

Solution First note that $\sin^2(x)/x = \sin(x)(\sin(x)/x)$. So using the given relationship we have,

$$\frac{\sin(2x)}{2} = \sin(x)\cos(x) \leq \sin(x)\frac{\sin(x)}{x} \leq \sin(x) \cdot 1 = \sin(x).$$

We have now been able to put the term $\sin^2(x)/x$ in between two functions. Looking at these functions they both go to the value of 0 as x goes to 0. By the squeezing principle we have that the function $\sin^2(x)/x$ will approach the value of 0 as x goes to 0.

18.5 Supplemental problems

1. Two trains start out ten miles apart on the same track and head toward each other, each going at five miles per hour. Between the two trains is a mathematical superfly who travels at a speed of ten miles per hour. The fly started on one train and is flying back and forth between the two trains. The fly is super in the sense that it can instantaneously turn around and start flying the other direction when it reaches one of the trains. Before the two trains collide the superfly will have made *infinitely* many trips back and forth between the two trains. How far will the fly have traveled? *Hint*: there is a very, very easy way to get the answer and a very, very hard way to get the answer; use the easy way.

2. Find the exact value of

$$\lim_{x\to 0} \frac{\sin(x+\pi)}{x}.$$

Scaling and magnitude have a nice relationship, namely that if we scale a vector by a value of c then we multiply the magnitude by $|c|$ (magnitude is *always* a non-negative number). The reason that this works is shown below.

$$\|c\vec{v}\| = \|\langle ca, cb\rangle\| = \sqrt{(ca)^2 + (cb)^2} = \sqrt{c^2(a^2+b^2)} = |c|\sqrt{a^2+b^2} = |c|\|\vec{v}\|$$

20.5 Working with direction

With a way to find magnitude we now turn to direction. This is trickier to get our finger on. What exactly is a direction? A good way to think about direction is as a *unit vector*. A unit vector is a vector with length one and so all of the important information about the vector is contained in the direction.

A useful fact is that every vector, besides the zero vector (i.e. $\langle 0,0\rangle$), can be represented in a *unique* way as a positive scalar times a unit vector. Namely, we have the following.

$$\vec{u} = \|\vec{u}\|\left(\frac{1}{\|\vec{u}\|}\vec{u}\right)$$

The important thing to note is that $(1/\|\vec{u}\|)\vec{u}$ is a unit vector. This follows from the argument just given about multiplying the magnitude of the vector by the same amount as you scale the vector.

Example 1 Find a unit vector in the same direction as

$$\langle 2, -5\rangle$$

Solution Proceeding with the idea just given we will divide this vector by its magnitude and get a unit vector pointing in the same direction. So we will get the following vector.

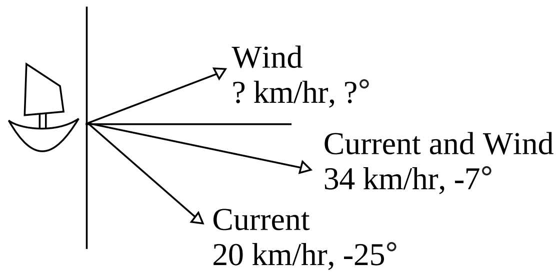
$$\frac{1}{\|\langle 2,-5\rangle\|}\langle 2,-5\rangle = \frac{1}{\sqrt{2^2+(-5)^2}}\langle 2,-5\rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}\right\rangle$$

There are two very important unit vectors that have been given names, these are called the standard unit vectors. They are $\mathbf{i} = \langle 1,0\rangle$ and $\mathbf{j} = \langle 0,1\rangle$. These are useful in giving another way to represent vectors in component form. Namely, we have the following,

$$\langle a,b\rangle = \langle a,0\rangle + \langle 0,b\rangle = a\langle 1,0\rangle + b\langle 0,1\rangle = a\mathbf{i} + b\mathbf{j}.$$

When you see a vector $\vec{u}$ in the form $a\mathbf{i}+b\mathbf{j}$, think of a as how much the vector is moving in the x direction and b as how much the vector is moving in the y direction. This is shown in the picture below.

at what speed and what direction would the boat travel? In other words, at what direction and what speed would the boat be traveling with only the wind? Round your answers to one decimal place. A badly drawn picture is shown below. *Hint*: to find the effect of the wind "subtract" the current from the combination of the current and wind.



Example 1 Find the dot product of two vectors the first of which has a magnitude of 12 and a direction of 53° and the second of which has a magnitude of 7 and a direction of 87° (where the angles are measured in standard position).

Solution We do not have the component form of these vectors and so we cannot directly apply the definition of the dot product. We could of course find the component form, but let us see if there is not a better way.

Notice that we can find the angle between the two vectors by taking the difference of their angles. In particular, the angle between these two vectors is $87° - 53° = 34°$. We already have the magnitudes of these vectors and so we can apply our new relationship for dot product and get the following.

$$\vec{u} \cdot \vec{v} = (12)(7)\cos(34°) \approx 69.63$$

Example 2 Find the angle between the vectors $\vec{u} = \langle 3, -8 \rangle$ and $\vec{v} = \langle -4, -2 \rangle$.

Solution This is a straightforward application of the dot product.

$$\begin{aligned}
\theta &= \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) \\
&= \arccos\left(\frac{(3)(-4) + (-8)(-2)}{\sqrt{(3)^2 + (-8)^2}\sqrt{(-4)^2 + (-2)^2}}\right) \\
&= \arccos\left(\frac{2}{\sqrt{365}}\right) \approx 83.99°
\end{aligned}$$

21.3 Orthogonal

Two vectors are perpendicular to one another if they meet at an angle of 90°. In particular if $\vec{u}$ and $\vec{v}$ are perpendicular we have the following,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(90°) = 0.$$

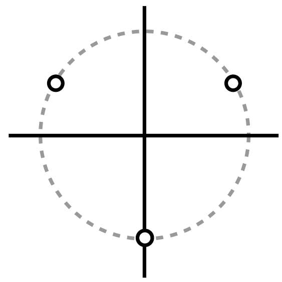
So we can use the dot product to test if two vectors are perpendicular. In general, we will say that two vectors whose dot product is zero are *orthogonal*. So two vectors that are perpendicular to one another are said to be orthogonal. By this convention we will say that $\vec{0}$ (i.e. the zero vector) is orthogonal to every vector.

(a) $\mathrm{proj}_{\vec{v}}(\vec{u}+\vec{w}) = \mathrm{proj}_{\vec{v}}(\vec{u}) + \mathrm{proj}_{\vec{v}}(\vec{w})$,

(b) $\mathrm{proj}_{\vec{v}}(a\vec{u}) = a\mathrm{proj}_{\vec{v}}(\vec{u})$.

[Note: any function that satisfies these two properties are *linear*. Linear functions form the backbone for much of mathematics.]

Graphically, these roots are around the unit circle as shown above.



23.8 Supplemental problems

1. Using De Moivre's formula find equations for $\cos(3x)$ and $\sin(3x)$ in terms of $\cos(x)$'s and $\sin(x)$'s.

2. Find $(1+\sqrt{3}i)^{15}$.

3. Find all the cube roots of $-4\sqrt{2}-4\sqrt{2}i$. Write your answers in trigonometric form.

4. Find all of the fifth roots of $-4+4i$. Write your answers in trigonometric form.

5. Using induction show that $1+2+\cdots+n = n(n+1)/2$. Verify this formula without induction by adding up the terms in forward and reverse order.

6. In an earlier homework assignment we found the following pattern:

$$\cos\left(\frac{\pi}{2^{(n+1)}}\right) = \frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}, \qquad \text{with } n \text{ square roots in total.}$$

   Using mathematical induction prove this relationship is true for $n = 1, 2, \ldots$. *Hint*: first verify it is true for the first case and then use the half angle formula for cosine to show that if it is true for one case then it is also true for the next case.