Infinite Sums

Let us start by explaining what an infinite sum is. Given a successor function $(a_n)_{n\in\mathbb{N}}$ *we can say that* $\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=1}^{m} a_n$ *if such a limit exists, and in that case we say the infinite sum or series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Now we shall state two basic properties of the infinite sum that have inherent usefulness:

1. $\sum_{n=n_0}^{\infty}(a_n \pm b_n) = \sum_{n=n_0}^{\infty} a_n + \sum_{n=n_0}^{\infty} b_n$
2. $\sum_{n=n_0}^{\infty} c.a_n = c.\sum_{n=n_0}^{\infty} a_n \ ,\forall_{c\in\mathbb{R}}$

Keeping in mind that our <u>*main goal is to determine if a series converges*</u>*, we will follow with three general observations:*

1. *If* $\sum_{n=n_0}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$ *(This obviously implies that if the limit of* a_n *does not exist or is different than zero then the series* $\sum_{n=n_0}^{\infty} a_n$ *does not converge).*
2. *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be successor functions so that* $a_n \le b_n$ *:*
   - *If* $\sum_{n=n_0}^{\infty} b_n$ *converges then* $\sum_{n=n_0}^{\infty} a_n$ *converges as well.*
   - *If* $\sum_{n=n_0}^{\infty} a_n$ *diverges then* $\sum_{n=n_0}^{\infty} b_n$ *diverges as well.*
3. *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be successor functions and* $\ell = \lim_{n\to\infty}\frac{a_n}{b_n}$*:*
   - *If* $\sum_{n=n_0}^{\infty} b_n$ *converges and* $\ell < +\infty$ *then* $\sum_{n=n_0}^{\infty} a_n$ *converges as well.*
   - *If* $\sum_{n=n_0}^{\infty} b_n$ *diverges and* $\ell > 0$ *then* $\sum_{n=n_0}^{\infty} a_n$ *diverges as well.*
   - *If* $\ell = 1$ *and* $a_n, b_n \ge 0$ *then both series have the same nature.*

With these observations in mind we will now see three known series:

- <u>*Geometric Series*</u>*:* $\sum_{n=p}^{\infty} r^n$ *if |r|<1 then the series converges to* $\frac{r^p}{1-r}$
- <u>*Mengoli Series*</u>*:* $\sum_{n=n_0}^{\infty}(a_n) = \sum_{n=n_0}^{\infty}(b_n - b_{n+1})$*, if this series is convergent then its sum is equal to* $b_{n_0} - \lim_{n\to\infty} b_n$ *,* $(a_n)_{n\in\mathbb{N}_0}$ *,* $(b_n)_{n\in\mathbb{N}_0}$ *successor functions.*
- <u>*Dirichlet Series*</u>*:* $\sum_{n=p}^{\infty}\frac{1}{n^\alpha}$ $\begin{cases} \alpha > 1 \text{ the series converges} \\ \alpha \le 1 \text{ the series diverges} \end{cases}$

Now we know three series and their convergence radius. Unfortunately most series don't fit in any of the types above. Nonetheless we can combine them with the first three observations and the criteria we will study next and use them to determine the nature of a series.

- <u>*D'Alambert's Criteria*</u>*: Let* $(a_n)_{n\in\mathbb{N}}$ *be a positive successor function and* $q = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

  $$\sum_{n=1}^{\infty}(a_n): \begin{cases} Converges, & if\ q < 1 \\ Diverges, & if\ q > 1 \\ Inconclusive, & if\ q = 1 \end{cases}$$

- <u>*Cauchy's Criteria*</u>*: Let* $(a_n)_{n\in\mathbb{N}}$ *be a positive successor function and* $q = \lim_{n\to\infty}\sqrt[n]{a_n}$

  $$\sum_{n=1}^{\infty}(a_n): \begin{cases} Converges, & if\ q < 1 \\ Diverges, & if\ q > 1 \\ Inconclusive, & if\ q = 1 \end{cases}$$

- <u>*Leibnitz Theorem*</u>*: Let* $(a_n)_{n\in\mathbb{N}}$ *be a successor function so that* $\exists_{b_n}: \ a_n = (-1)^n b_n$*. If* $b_n \ge 0$ *,* $\lim_{n\to\infty} b_n = 0$ *and* $b_n \ge b_{n+1}$*, then we can conclude that* $\sum_{n=n_0}^{\infty}(a_n)$ *is convergent.*

  Thus we conclude our introductory approach to infinite sums.