PERMUTATION & COMBINATION

1. Factorial notation -

The continuous product of first n natural numbers is called factorial

i.e. $|\underline{n}$ or $n! = 1.\ 2.\ 3........(n-1).n$

$n! = n(n-1)! = n(n-1)(n-2)!$ & so on

or $n(n-1).........(n-r+1) = \dfrac{n!}{(n-r)!}$

Here $0! = 1$ and $(-n)! =$ meaningless.

2. Fundamental principle of counting -

(i) **Addition rule :** If there are two operations such that they can be done independently in m and n ways respectively, then either (any one) of these two operations can be done by (m + n) ways.

Addition ⇒ OR (or) Option

(ii) **Multiplication rule :** Let there are two tasks of an operation and if these two tasks can be performed in m and n different number of ways respectively, then the two tasks together can be done in m × n ways.

Multiplication ⇒ And (or) Condition

(iii) **Bijection Rule :** Number of favourable cases

= Total number of cases

− Unfavourable number of cases.

3. Permutations (Arrangement of objects) -

(i) The number of permutations of n different things taken r at a time is $^{n}p_{r} = \dfrac{n!}{(n-r)!}$

(ii) The number of permutations of n dissimilar things taken all at a time is $^{n}p_{n} = n!$

(iii) The number of permutations of n distinct objects taken r at a time, when repetition of objects is allowed is n^{r}.

(iv) If out of n objects, 'a' are alike of one kind, 'b' are alike of second kind and 'c' are alike of third kind and the rest distinct, then the number of ways of permuting the n objects is $\dfrac{n!}{a!\,b!\,c!}$

4. Restricted Permutations -

(i) The number of permutations of n dissimilar things taken r at a time, when m particular things always occupy definite places $= {}^{n-m}p_{r-m}$

(ii) The number of permutations of n different things taken r at a time, when m particular things are always to be excluded (included)

$= {}^{n-m}P_{r}\ ({}^{n-m}C_{r-m} \times r!)$

5. Circular Permutations -

When clockwise & anticlockwise orders are treated as different.

(i) The number of circular permutations of n different things taken r at a time $\dfrac{^{n}P_{r}}{r}$

(ii) The number of circular permutations of n different things taken altogether $\dfrac{^{n}P_{n}}{n} = (n-1)!$

When clockwise & anticlockwise orders are treated as same.

(i) The number of circular permutations of n different things taken r at a time $\dfrac{^{n}P_{r}}{2r}$

(ii) The number of circular permutations of n different things taken all together $\dfrac{^{n}P_{n}}{2n} = \dfrac{1}{2}(n-1)!$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A} \quad \Rightarrow \tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

(iv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 4\sin(60^0 - \theta)\sin(60^0 + \theta)\sin\theta$

$= \sin\theta\,(2\cos\theta - 1)(2\cos\theta + 1)$

(v) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$= 4\cos(60^0 - \theta)\cos(60^0 + \theta)\cos\theta$

$= \cos\theta\,(1 - 2\sin\theta)(1 + 2\sin\theta)$

(vi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$= \tan(60^0 - A)\tan(60^0 + A)\tan A$

(vii) $\sin A/2 = \sqrt{\dfrac{1-\cos A}{2}}$

(viii) $\cos A/2 = \sqrt{\dfrac{1+\cos A}{2}}$

(ix) $\tan A/2 = \sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A}$, $A \neq (2n+1)\pi$

9. Maximum and minimum value of the expression :

$a\cos\theta + b\sin\theta$

Maximum (greatest) Value $= \sqrt{a^2 + b^2}$

Minimum (Least) value $= -\sqrt{a^2 + b^2}$

10. Conditional trigonometric identities :

If A, B, C are angles of triangle i.e. $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

i.e. $\sum \sin 2A = 4 \prod (\sin A)$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iii) $\sin A + \sin B + \sin C = 4\cos A/2 \cos B/2 \cos C/2$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin A/2 \sin B/2 \sin C/2$

(v) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \cos C$

(vi) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

(vii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(viii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

(ix) $\sum \tan A/2 \tan B/2 = 1$

(x) $\sum \cot A \cot B = 1$

(xi) $\sum \cot A/2 = \prod \cot A/2$

11. Some useful series :

(i) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots. + \text{to nterms}$

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left[\frac{n\beta}{2}\right]}{\sin\frac{\beta}{2}}, \ \beta \neq 2n\pi$$

(ii) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots. + \text{to nterms}$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \ \beta \neq 2n\pi$$

(iii) $\cos\alpha\,.\cos 2\alpha\,.\,\cos 2^2\alpha \ \ldots.\cos(2^{n-1}\alpha) = \dfrac{\sin 2^n\alpha}{2^n \sin\alpha}, \ \alpha \neq n\pi$

$= 1, \ \alpha = 2k\pi$

$= -1, \ \alpha = (2k+1)\pi$

Preview from Notesale.co.uk Page 20 of 93

6. Inverse trigonometric ratios of multiple angles

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$, if $-1 \le x \le 1$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$, if $-1 \le x \le 1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(iv) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

(v) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

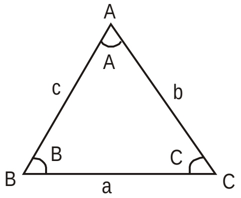
(vi) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

PROPERTIES & SOLUTION OF TRIANGLE

Properties of triangle :

1. A triangle has three sides and three angles.

In any ΔABC, we write $BC = a$, $AB = c$, $AC = b$



and $\angle BAC = \angle A$, $\angle ABC = \angle B$, $\angle ACB = \angle C$

2. In ΔABC :

(i) $A + B + C = \pi$

(ii) $a + b > c$, $b + c > a$, $c + a > b$

(iii) $a > 0$, $b > 0$, $c > 0$

3. Sine formula :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k(\text{say})$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k\ (\text{say})$$

4. Cosine formula :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

4. Position of a point with respect to a straight line :
The line $L(x_i, y_i)$ i = 1, 2 will be of same sign or of opposite sign according to the point $A(x_1, y_1)$ & B (x_2, y_2) lie on same side or on opposite side of L (x, y) respectively.

5. Equation of a line parallel (or perpendicular) to the line
$ax + by + c = 0$ is $ax + by + c' = 0$ (or $bx - ay + \lambda = 0$)

6. Equation of st. lines through (x_1,y_1) making an angle α with $y = mx + c$ is

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

7. length of perpendicular from (x_1, y_1) on $ax + by + c = 0$

is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

8. Distance between two parallel lines $ax + by + c_i = 0$,

i = 1, 2 is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

9. Condition of concurrency for three straight lines
$L_i \equiv a_i x + b_i y + c_i = 0$, i = 1, 2, 3 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

10. Equation of bisectors of angles between two lines :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

11. Family of straight lines :
The general equation of family of straight line will be written in one parameter
The equation of straight line which passes through point of intersection of two given lines L_1 and L_2 can be taken as $L_1 + \lambda L_2 = 0$

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12. Homogeneous equation : If $y = m_1x$ and $y = m_2x$ be the two equations
represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$

13. General equation of second degree :
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight line if $\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

If $y = m_1x + c$ & $y = m_2x + c$ represents two straight lines then $m_1 + m_2 = \dfrac{-2h}{b}$, $m_1m_2 = \dfrac{a}{b}$.

14. Angle between pair of straight lines :
The angle between the lines represented by
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ or $ax^2 + 2hxy + by^2 = 0$

is $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{(a + b)}\right|$

(i) The two lines given by $ax^2 + 2hxy + by^2 = 0$ are
  (a) Parallel and coincident iff $h^2 - ab = 0$
  (b) Perpendicular iff $a + b = 0$
(ii) The two line given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are
  (a) Parallel if $h^2 - ab = 0$ & $af^2 = bg^2$
  (b) Perpendicular iff $a + b = 0$
  (c) Coincident iff $g^2 - ac = 0$

13. Combined equation of angle bisector of the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

12. Equation of polar of the point (x_1, y_1) w.r.t. the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is given by } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0 \text{ i.e. } T = 0.$$

The pole of the line $\ell x + my + n = 0$ w.r.t. the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \left(\frac{-a^2\ell}{n}, \frac{-b^2n}{n}\right).$$

13. Eccentric angles of the extremities of latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.

14. (i) Equation of the diameter bisecting the chords of slope in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = -\frac{b^2}{a^2m}x$$

(ii) **Conjugate Diameters :** The straight lines $y = m_1x$, $y = m_2x$ are conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } m_1m_2 = -\frac{b^2}{a^2}.$$

(iii) Properties of conjugate diameters :

(a) If CP and CQ be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$CP^2 + CQ^2 = a^2 + b^2$

(b) If θ and ϕ are the eccentric angles of the extremities of two conjugate diameters, then

$$\theta - \phi = \pm\frac{\pi}{2}$$

(c) If CP, CQ be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S, S' be two foci of the ellipse, then $SP.S'P = CQ^2$

(d) The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

15. The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axis i.e. 4ab.

16. Length of subtangent and subnormal at $p(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2}{x_1} - x_1 \ \&\ (1 - e^2)\, x_1$$

2. **Special form of hyperbola :**

If the centre of hyperbola is (h, k) and axes are parallel to the co-ordinate axes, then its equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

3. **Parametric equations of hyperbola :**

The equations $x = a \sec \phi$ and $y = b \tan \phi$ are known as the parametric equations of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Preview from Notesale.co.uk Page 44 of 93

4. **Position of a point and a line w.r.t. a hyperbola :**

The point (x_1, y_1) lies inside, on or outside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is +ve, zero or −ve.

The line $y = mx + c$ does not intersect, touches, intersect the hyperbola according as $c^2 <, =, > a^2m^2 - b^2$.

5. **Equations of tangents in different forms :**

(a) **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(b) **Parametric form :** The equation of tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } (a \sec \phi, b \tan \phi) \text{ is}$$

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1.$$

(c) **Slope form :** The equations of tangents of slope m to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right).$$

6. Equation of pair of tangents from (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $SS_1 = T^2$

7. **Equations of normals in different forms :**

(a) **Point form :** The equation of normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1) \text{ is } \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2.$$

(b) **Parametric form :** The equation of normal at $(a \sec \theta, b \tan \theta)$ to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } ax \cos \theta + by \cot \theta = a^2 + b^2$$

* The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

* The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\frac{y^2}{b^2}$ or $2\sec^{-1} e$.

* A hyperbola and its conjugate hyperbola have the same asymptotes.

* The bisector of the angles between the asymptotes are the coordinate axes.

* Equation of hyperbola – Equation of asymptotes = Equation of asymptotes – Equation of conjugate hyperbola = constant.

16. Rectangular or Equilateral Hyperbola :

* A hyperbola for which a = b is said to be rectangular hyperbola, its equation is $x^2 - y^2 = a^2$

* $xy = c^2$ represents a rectangular hyperbola with asymptotes $x = 0$, $y = 0$.

* Eccentricity of rectangular hyperbola is $\sqrt{2}$ and angle between asymptotes of rectangular hyperbola is 90º.

* Parametric equation of the hyperbola $xy = c^2$ are

  $x = ct$, $y = \frac{c}{t}$, where t is a parameter.

* Equation of chord joining t_1, t_2 on $xy = c^2$ is

  $x + y\, t_1t_2 = c(t_1 + t_2)$

Preview from Notesale.co.uk Page 46 of 93

* Equation of tangent at (x_1, y_1) to $xy = c^2$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

* Equation of tangent at t is $x + yt^2 = 2ct$

* Equation of normal at (x_1, y_1) to $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$

* Equation of normal at t on $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

  (This results shows that four normal can be drawn from a point to the hyperbola $xy = c^2$)

* If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.

* Equation of chord of the hyperbola $xy = c^2$ whose middle point is given is $T = S_1$

* Point of intersection of tangents at t_1 & t_2 to the hyperbola $xy = c^2$ is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$

Properties :

(i) $x - 1 < [x] \le x$

(ii) $[x + I] = [x] + I$

$[x + y] \ne [x] + [y]$

(iii) $[x] + [-x] = 0, x \in I$

$= -1, x \notin I$

(iv) $[x] = I$, where I is an integer $x \in [I, I + 1)$

(v) $[x] \ge I, x \in [I, \infty)$

(vi) $[x] \le I, x \in (-\infty, I + 1]$

(vii) $[x] > I, [x] \ge I + 1, x \in [I + 1, \infty)$

(viii) $[x] < I, [x] \le I - 1, x \in (-\infty, I)$

4. Fractional part function :

$f(x) = \{x\}$ = difference between number & its integral part

$= x - [x]$.

Properties :

(i) $\{x\}, x \in [0, 1)$

(ii) $\{x + I\} = \{x\}$

$\{x + y\} \ne \{x\} + \{y\}$

(iii) $\{x\} + \{-x\} = 0, \quad x \in I$

$= 1, x \notin I$

(iv) $[\{x\}] = 0, \{\{x\}\} = \{x\}, \{[x]\} = 0$

5. Signum function :

$$f(x) = \text{sgn}(x) = \begin{cases} -1 & , \quad x \in R^- \\ 0 & , \quad x = 0 \\ 1 & , \quad x \in R^+ \end{cases}$$

$$\text{or} \quad f(x) = \frac{|x|}{x}, \quad x \ne 0$$

$$= 0, \quad x = 0$$

6. Definition :

Let A and B be two given sets and if each element $a \in A$ is associated with a unique element $b \in B$ under a rule f, then this relation (mapping) is called a function.

Graphically - no vertical line should intersect the graph of the function more than once.

Here set A is called domain and set of all f images of the elements of A is called range.

i.e., Domain = All possible values of x for which f(x) exists.

Range = For all values of x, all possible values of f(x).

Table : Domain and Range of some standard functions -

Functions	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	(K)
Reciprocal function $\frac{1}{x}$	R_0	R_0
x^2, $\|x\|$ (modulus function)	R	$R^+ \cup \{x\}$
x^3, $x\|x\|$	R	R
Signum function $\frac{\|x\|}{x}$	R	{-1, 0, 1}
$x + \|x\|$	R	$R^+ \cup \{x\}$
$x - \|x\|$	R	$R^- \cup \{x\}$
[x] (greatest integer function)	R	1
x - {x}	R	[0, 1]
$\sqrt{x}$	$[0, \infty)$	$[0, \infty]$
a^x (exponential function)	R	R^+
log x (logarithmic function)	R^+	R

Preview from Notesale.co.uk Page 55 of 93

Function	Period
sin x, cos x sec x, cosec x	2π
tan x, cot x	π
sin (x/3)	6π
tan 4x	$\pi/4$
cos 2πx	1
|cos x|	π
$\sin^4 x + \cos^4 x$	$\pi/2$
$2\cos\left(\frac{x-\pi}{3}\right)$	6π
$\sin 3x + \cos^3 x$	$2\pi/3$
$\sin^3 x + \cos^4 x$	2π
$\frac{\sin x}{\sin 5x}$	2π
$\tan^2 x - \cot^2 x$	π
x − [x]	1
[x]	1

NON PERIODIC FUNCTIONS :

$\sqrt{x}$, x^2, x^3, 5

$\cos x^2$

x + sin x

x cos x

$\cos\sqrt{x}$

15. Composite function :

If $f : X \to Y$ and $g : Y \to Z$ are two function, then the composite function of f and g, $gof : X \to Z$ will be defined as $gof(x) = g(f(x)),\ \forall\ x \in X$

In general $gof \neq fog$

If both f and g are bijective function, then so is gof.

Illy $\frac{d^3y}{dx^3} = \frac{d^2}{dx^2}(f'(x))$ etc......

Thus, This process can be continued and we can obtain derivatives of higher order

Note : To obtain higher order derivative of parametric functions we use chain rule

i.e. if $x = 2t,\ y = t^2$

$$\Rightarrow \quad \frac{dy}{dx} = t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(t) = 1.\frac{dt}{dx} = \frac{1}{t}$$

(b) If $y = (ax + b)^m$ $m \notin I$, then

$y_n = m(m-1)(m-2) \ldots\ldots (m-n+1)(ax + b)^{m-n}.a^n$

(c) If $m \in I$, then

$y_m = m!\ a^m$ and $y_{m+1} = 0$

(d) If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n n!}{(ax+b)^{n+1}} a^n$

(e) If $y = \log(ax + b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} a^n$

(f) If $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

If $y = \cos(ax + b)$, then $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$

6. n^th Derivatives of Some Functions :

(i) $\frac{d^n}{dx^n}(x^n) = n!$

(ii) $\frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right)$

(iii) $\frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{\pi n}{2}\right)$

(iv) $\frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}$

(v) $\frac{d^n}{dx^n}(\log x) = (-1)^{n-1}(n-1)!\ x^{-n}$

NOTE : If $u = g(x)$ is such that $g'(x) = K$ (constant)

then $\frac{d^n}{dx^n} f(g(x)) = K^n \left[\frac{d^n}{du^n} f(u)\right]_{u=g(x)}$

11. If the normal is parallel to x-axis ⇒ $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0.$

12. If the normal is perpendicular to x-axis ⇒ $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0.$

13. If normal is equally inclined from both the axes or cuts equal intercept then $\left(\frac{dy}{dx}\right) = \pm 1.$

14. The equation of the normal to the curve y = f(x) at a point (x_1, y_1) is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}(x - x_1)$$

15. Length of intercept made on axes by the normal :

$$x - \text{intercept} = x_1 + y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$y - \text{intercept} = y_1 + x_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

16. Length of perpendicular from origin to normal :

$$= \left| \frac{x_1 + y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right|$$

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17. Angle of intersection of the two curves :

$$\tan\theta = \pm \frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1 - \left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2}$$

where $\left(\frac{dy}{dx}\right)_1$ is the slope of first curve & $\left(\frac{dy}{dx}\right)_2$ of second. If both curves intersect orthogonally then $\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1$

18. Length of tangent, normal, subtangent & subnormal :

$$\text{Length of tangent} = \left| \frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right|$$

$$\text{Length of normal} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\text{Length of sub-tangent} = \frac{y}{dy/dx}$$

$$\text{Length of sub-normal} = y\frac{dy}{dx}$$

INDEFINITE INTEGRATION

1. (i) If $\frac{d}{dx}F(x) = f(x)$, then $\int f(x)dx = F(x) + c$

Here $\int\{\ \}dx$ is the notation of integration, f(x) is the integrand, c is any real no. (integrating constant)

(ii) $\frac{d}{dx}\int f(x)dx = f(x)$

(iii) $\int f'(x)dx = f(x) + c,\ c \in R$

(iv) $\int k\ f(x)dx = k\int f(x)dx$

(v) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

2. **FUNDAMENTAL FORMULAE :**

Function	**Integration**
$\int x^n dx$	$\frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\int (ax+b)^n dx$	$\frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1$
$\int \frac{1}{x}dx$	$\log\lvert x\rvert + c$
$\int \frac{1}{ax+b}dx$	$\frac{1}{a}(\log\lvert ax + b\rvert) + c$
$\int e^x dx$	$e^x + c$
$\int a^x dx$	$\frac{a^x}{\log_e a} + c$
$\int \sin x\ dx$	$-\cos x + c$

Function	**Integration**
$\int \cos x\ dx$	$\sin x + c$
$\int \sec^2 x\ dx$	$\tan x + c$
$\int \operatorname{cosec}^2 x\ dx$	$-\cot x + c$
$\int \sec x \tan x\ dx$	$\sec x + c$
$\int \operatorname{cosec} x \cot x dx$	$-\operatorname{cosec} x + c$
$\int \tan x\ dx$	$-\log\lvert\cos x\rvert + c = \log\lvert\sec x\rvert + c$
$\int \cot x\ dx$	$\log\lvert\sin x\rvert + c = -\log\lvert\operatorname{cosec} x\rvert + c$
$\int \sec x\ dx$	$\log\lvert\sec x + \tan x\rvert + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$
$\int \operatorname{cosec} x\ dx$	$\log\lvert\operatorname{cosec} x - \cot x\rvert + c = \log \tan\frac{x}{2} + c$
$\int \frac{dx}{\sqrt{1-x^2}}$	$\sin^{-1} x + c = -\cos^{-1} x + c$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$
$\int \frac{dx}{1+x^2}$	$\tan^{-1} x + c = -\cot^{-1} x + c$
$\int \frac{dx}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a} + c = \frac{-1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

(iii) (a) $\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + c$

(b) $\int e^{mx}[mf(x)+f'(x)]\,dx = e^{mx} f(x) + c$

(c) $\int e^{mx}\left[f(x)+\frac{f'(x)}{m}\right]dx = \frac{e^{mx}f(x)}{m} + c.$

(iv) $\int [xf'(x)+f(x)]\,dx = x\,f(x) + c.$

NOTE : Breaking (iii) & (iv) integral into two integrals. Integrate one integral by parts and keeping other integral as it is, by doing so we get the result (integral).

(v) $\int e^{ax}\sin bx\,dx$ and $\int e^{ax}\sin(bx+c)\,dx$

$$= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + k \text{ and}$$

$$\frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)] + k_1$$

(vi) $\int e^{ax}\cos bx\,dx$ and $\int e^{ax}\cos(bx+c)\,dx$

$$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + k$$

and $\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + k_1.$

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5. INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS USING PARTIAL FRACTION :

Every Rational fun. may be represented in the form $\frac{P(x)}{Q(x)}$, where P(x), Q(x) are polynomials.

If degree of numerator is less than that of denominator, the rational fun. is said to be proper other wise it is improper. If deg (num.) ≥ deg(deno.) apply division rule

i.e. $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, for integrating $\frac{r(x)}{g(x)}$, resolve the fraction into partial factors. The following table illustrate the method.

Types of proper rational functions	**Types of partial fractions**
$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$, a, b, c are distinct	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where $x^2 + bx + c$ can not be factorised	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where $x^2 + ax + b$, $x^2 + cx + d$ can not be factorised	$\frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d}$

(B) $\int \sin^m x \cos^m x dx$.

- (i) if m is odd put cos x = t
- (ii) if m is even put sin x = t
- (iii) if m & n both odd put sin x or cos x as t
- (iv) if m & n both even use the formula of $\sin^2 x$ & $\cos^2 x$
- (v) if m & n rational no. & $\frac{m+n-2}{2}$ is –ve integer put tan x = t

8. INTEGRATION BY SUCCESSIVE REDUCTION (REDUCTION FORMULA) :

Function	Integration
$\int x^n e^{ax} dx$, n ∈ N	$\frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$ where $I_{n-1} = \int x^{n-1} e^{ax} dx$
$\int x^n \sin x \, dx$	$-x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}$
$\int \sin^n x \, dx$	$-\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$
$\int \cos^n x \, dx$	$\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$
$\int \tan^n x \, dx$	$\frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$
$\int \cot^n x \, dx$	$-\frac{(\cot x)^{n-1}}{n-1} - I_{n-2}$

$\int \sec^n x \, dx$	$\frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$
$\int \mathrm{cosec}^n x \, dx$	$-\frac{\mathrm{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$
$\int \sin^m x \cos^n x \, dx$	$\frac{\cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,\,n-2}}{(m+n)}$ $-\sin^{m-1}x \cos^{n+1}x + (m-1) I_{m-2,n}$

NOTE : These formulae are specifically useful when m & n are both even nos.

VECTORS

1. Types of vectors :

(a) Zero or null vector : A vector whose magnitude is zero is called zero or null vector.

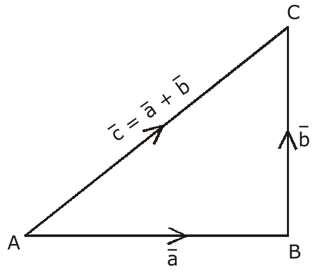
(b) Unit vector : $\hat{a} = \dfrac{\vec{a}}{|a|} = \dfrac{\text{Vector a}}{\text{Magnitude of a}}$

(c) Equal vector : Two vectors $\bar{a}$ and $\bar{b}$ are said to be equal if |a| = |b| and they have the same direction.

Preview from Notesale.co.uk Page 85 of 93

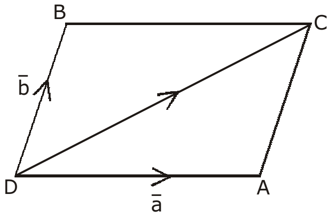
2. Triangle law of addition : $\overline{AB} + \overline{BC} = \overline{AC}$

$$\bar{c} = \bar{a} + \bar{b}$$



3. Parallelogram law of addition : $\overline{OA} + \overline{OB} = \overline{OC}$

$$\bar{a} + \bar{b} = \bar{c}$$



where OC is a diagonal of the parallelogram OABC

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4. Vectors in terms of position vectors of end points -

$\overline{AB} = \overline{OB} - \overline{OA}$ = Position vector of B – position vector of A

i.e. any vector = p.v. of terminal pt – p.v. of initial pt.

5. Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then m$\vec{a}$ is a vector and magnitude of m$\vec{a}$ = m|a|

and if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$

6. Distance between two points :

Distance between points A(x_1, y_1, z_1) and B(x_2, y_2, z_2)

$$= \text{Magnitude of } \vec{AB}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

7. Position vector of a dividing point :

(i) If A($\bar{a}$) & B($\vec{b}$) be two distinct pts, the p.v. $\bar{c}$ of the point C dividing [AB] in ratio $m_1 : m_2$ is given by

$$\vec{c} = \frac{m_1\vec{b} + m_2\vec{a}}{m_1 + m_2}$$

(ii) p.v. of the mid point of [AB] is $\frac{1}{2}$ [p.v. of A + p.v. of B]

(iii) If point C divides AB in the ratio $m_1 : m_2$ externally,

then p.v. of C is $\bar{c} = \dfrac{m_1\bar{b} - m_2\bar{a}}{m_1 - m_2}$