**Question 3.** *Verify that for all $n \geq 1$, the sum of the squares of the first $2n$ positive integers is given by the formula*

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

**Solution.**

For any integer $n \geq 1$, let $P_n$ be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}.$$

<u>Base Case.</u> The statement $P_1$ says that

$$1^2 + 2^2 = \frac{(1)(2(1)+1)(4(1)+1)}{3} = \frac{3(5)}{3} = 5,$$

which is true.

<u>Inductive Step.</u> Fix $k \geq 1$, and suppose that $P_k$ holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}.$$



It remains to show that $P_{k+1}$ holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}.$$

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 &= 1^2 + 2^2 + 3^2 + \cdots + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 && \text{(by } P_k\text{)} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(8k^2+6k+1) + 3(4k^2+4k+1) + 3(4k^2+8k+4)}{3} \\
&= \frac{(8k^3+6k^2+k) + (12k^2+12k+3) + (12k^2+24k+12)}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3}
\end{aligned}$$

On the other side of $P_{k+1}$,

$$\frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3} = \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$$