flat for small values of x. Therefore it can be approximated to 0. Hence, the expression simplifies to:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(x) + \cos(x)\delta x - sin(x)}{\delta x} = \cos(x)$$

Note: *these approximations are not very scientific, and further on methods of doing it properly will be discussed.*

For a function to be differentiable, it has to be continuous and it cannot have any "hard edges" like $|x|$. There are many different notations of derivatives. The most popular for first derivative include:

$$\frac{dy}{dx} = f'(x) = f^{(1)}(x) = \dot{x}$$

For second, and further:

$$\frac{d^2y}{dx^2} = f''(x) = f^{(2)}(x) = \ddot{x}$$

Note: *notation always should be a slave of a mathematician!*

In practice, first principles aren't very useful. The most common derivatives (that everyone needs to know by hard) include:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\mathrm{arcos}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
e^x	e^x
$\log_a x$	$\frac{1}{x\ln(a)}$

Exercise 2.1

By expressing a and b in a 3D Cartesian basis, e.g. $\underline{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, and $\underline{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$, and multiplying out the product show that the scalar product may be written in the form:

$$\underline{a} \cdot \underline{b} = (a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) \cdot (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) =$$

$$a_x\hat{x} \cdot b_x\hat{x} + a_x\hat{x} \cdot b_y\hat{y} + a_x\hat{x} \cdot b_z\hat{z} +$$

$$a_y\hat{y} \cdot b_x\hat{x} + a_y\hat{y} \cdot b_y\hat{y} + a_y\hat{y} \cdot b_z\hat{z} +$$

$$a_z\hat{z} \cdot b_x\hat{x} + a_z\hat{z} \cdot b_y\hat{y} + a_z\hat{z} \cdot b_z\hat{z}$$

Since $\delta_{\hat{x}\hat{y}} = \delta_{\hat{x}\hat{z}} = \delta_{\hat{y}\hat{z}} = 0$ (see Kronecker delta notation) the expression simplifies to:

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$$

Exercise 2.2

You are given the vectors $\underline{a} = 5.0\hat{x} - 6.5\hat{y}$ and $\underline{b} = -3.5\hat{x} + 7.0\hat{y}$. A third vector $\underline{c}$, lies in the xy-plane. Vector $\underline{c}$ is perpendicular to vector $\underline{a}$ and the scalar product of $\underline{c}$ with $\underline{b}$ is 15.0. Find the x and y-components of the vector $\underline{c}$ and the angle between vectors $\underline{b}$ and $\underline{a}$.

$$\begin{cases} \underline{a} \cdot \underline{c} = 0 = 5c_x - 6.5c_y \\ \underline{b} \cdot \underline{c} = 15 = -3.5c_x + 7c_y \end{cases}$$

Solving simultaneous equations produces:

$$\begin{cases} c_x = 8 \\ c_y = 6.12 \end{cases}$$

The angle between vectors $\underline{b}$ and $\underline{a}$ is given by:

$$\theta = \arccos \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = 2.95\ radians$$

Exercise 2.3

Show that $\frac{dv^2}{dt} = 2\underline{a} \cdot \underline{v}$ where $\underline{v}$ is the velocity, and $\underline{a}$ is the acceleration. Since $v^2 = \underline{v} \cdot \underline{v}$:

$$\frac{d}{dt}(\underline{v} \cdot \underline{v}) = \frac{d\underline{v}}{dt} \cdot \underline{v} + \underline{v} \cdot \frac{d\underline{v}}{dt} = \underline{a} \cdot \underline{v} + \underline{v} \cdot \underline{a} = 2\underline{a} \cdot \underline{v}$$

Example

Transform $\frac{df}{dx}+\frac{df}{dy}=0$ **into polar coordinates:**

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}cos\theta+\frac{\partial f}{\partial y}sin\theta$$

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x}(-rsin\theta)+\frac{\partial f}{\partial y}rcos\theta$$

Solving for $\frac{\partial f}{\partial x}$ **and** $\frac{\partial f}{\partial y}$ **produces:**

$$(cos\theta+sin\theta)\frac{\partial f}{\partial r}+\left(\frac{cos\theta}{r}-\frac{sin\theta}{r}\right)\frac{\partial f}{\partial \theta}=0$$

Stationary points

Stationary points in functions with two variables ($f(x,y)$**) occur when:**

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$$

To classify them, first compute:

$$\Delta=\left(f_{xy}(a,b)\right)^2-f_{xx}(a,b)f_{yy}(a,b)$$

Where a **and** b **are the coordinates of stationary point.**

if $\Delta>0$**, saddle point**

if $\Delta<0$ **and** $f_{xx}(a,b)<0$**, maxima**

if $\Delta<0$ **and** $f_{xx}(a,b)>0$**, minima**

$if\ \Delta=0$ **it can be both, the method does not tell**

Note: in case of classification of maxima and minima, there is no difference whether $f_{xx}(a,b)$ or $f_{yy}(a,b)$ is used.

12. VECTORS 2

The vector product (or cross product) takes two vectors $\underline{a}$ **and** $\underline{b}$ **and produces a third vector** $\underline{c}$ **in the direction which is perpendicular to the plane containing** $\underline{a}$ **and** $\underline{b}$ **. to determine the sense of resultant vector, right hand rule is used (if you curl the fingers of your right hand in the same direction of rotation as when turning from the first vector to the**

the voltage and current are in phase, i.e. the two waveforms are exactly the same apart from the amplitude.

i. $\omega = 2\pi f = 6.28 \times 10^6 rad \times s^{-1}$

$\tilde{V} = 8 \times 10^{-3} e^{i\omega t}\ Volts$

ii. $Z = 24 + \frac{1}{1 \times 6.28 \times 10^6 \times 5 \times 10^{-10}} = 24 - 318.3i = 319.2e^{-1.496i}$

iii. $I = Re\left[\frac{8e^{i\omega t} \times 10^{-3}}{319.2e^{-1.496i}}\right] = 2.506 \times 10^{-5}\cos(\omega t + 1.496)\ Ampers$

iv. Since current is shifted by approximately $\frac{\pi}{2}$ as in b), it can be concluded that capacitor dominates the circuit

Exercise 4.7

In a series RC (resistor–capacitor) circuit the voltage across the resistor is given by:

$$V_R = Acos(\omega t)$$

And the voltage across the capacitor:

$$V_C = Bsin(\omega t)$$

Find the combined voltage in the form of $V = Ccos(\omega t + S)$.

$$V = V_R + V_C = Acos(\omega t) + Bcos(\omega t) = Re(Ae^{i\omega t}) - (iBe^{i\omega t})$$

The difference cab be rewritten as a product of two complex numbers:

$$Re((A - iB)e^{i\omega t})$$

Converting first one to polar form (minus sign because the number is in fourth quadrant):

$$Re\left(\sqrt{A^2 + B^2}e^{-\arctan\left(\frac{B}{A}\right)} \times e^{i\omega t}\right) = \sqrt{A^2 + B^2}\cos\left(\omega t - \arctan\left(\frac{B}{A}\right)\right)$$

14. POWER SERIES

Arithmetic progressions

In arithmetic series there is a constant difference (denoted as d) between consecutive terms:

$$u_{n+1} = u_n + d = u_1 + (n-1)d$$

Where nth terms is written as u_n. The first term is also often denoted as a. The sum of such series is just an average of first and last term multiplied by the number of terms:

$$S_n = N\frac{u_n + u_1}{2}$$

Finite geometric series

In this series, each term is a constant multiple of the previous last one.

$$u_n = u_{n-1}r = ar^{n-1}$$

The sum of a finite geometric series is:

$$S_n = a + ar + ar^2 + \cdots + ar^n$$

Multiplying by r (the common ratio) produces:

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtracting these two series:

$$S_n - rS_n = a - ar^n \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

Infinite geometric progression

$$S_n = \sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r}$$

The answer depends on the value of the common ratio:

- if $|r| > 1$, r^n diverges and whole sum diverges,
- if $r = \pm 1$, the sum also diverges (write down consecutive terms to see it),
- if $|r| < 1$, the sum converges and becomes:

$$S = \frac{a}{1-r}$$

Useful rules:

Integral of the sum of functions is the sum of integrals of those:

$$\int f(x) + g(x) = \int f(x) + \int g(x)$$

Constant multiple can be taken outside the integral:

$$\int cf(x) = c \int f(x)$$

Reversing limits gives negative sign:

$$\int_a^b f(x) = -\int_b^a f(x)$$

Definite integral can be rewritten as the sum of definite integrals:

$$\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x), \qquad a < c < b$$

Definite integral with same limits equals 0:

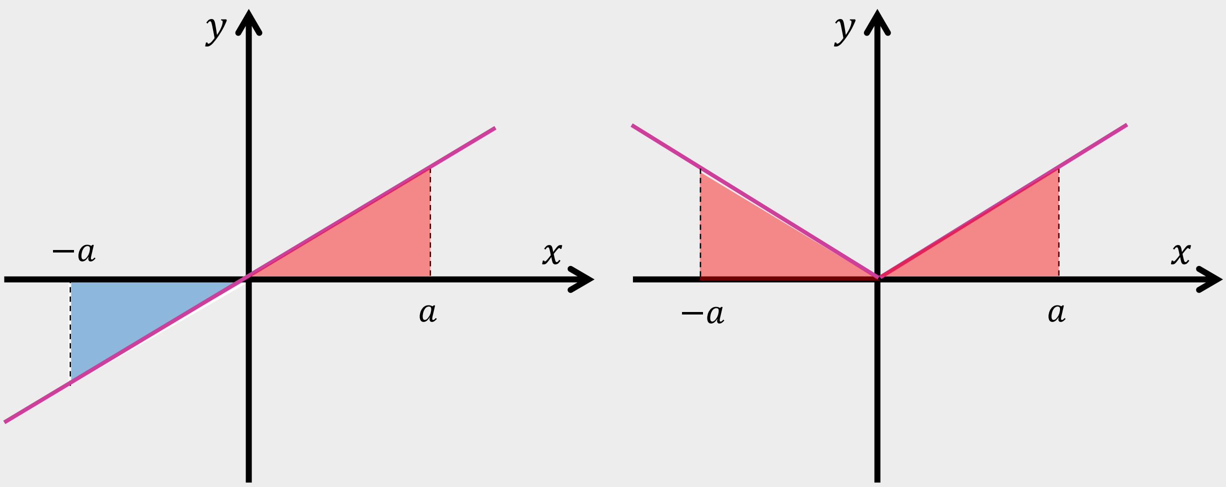
$$\int_a^a f(x) = 0$$

For odd functions:

$$\int_{-a}^{a} f(x) = 0, \qquad f(x) = -f(-x)$$

For even functions:

$$\int_{-a}^{a} f(x) = 2\int_0^a f(x), \qquad f(x) = f(-x)$$



Standard integrals:

$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1} + c$
$\sin(x)$	$-\cos(x) + c$
$\cos(x)$	$\sin(x) + c$
$\cosh(x)$	$\sinh(x) + c$
$\sinh(x)$	$\cosh(x) + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$

Perfect integrals

These are trivial to prove using integration by substitution, that will be introduced shortly:

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)f(x)dx = \frac{1}{2}(f(x))^2 + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2(f(x))^{\frac{1}{2}} + c$$

Note: integration is probably the hardest bit of this course, everyone struggles with it, the only way to improve is to exercise and learn how to recognize patterns.

Example

From equation of motion of a particle going from $x = a$ **to** $x = b$:

$$F(x) = m\frac{dv}{dt}$$

Show that the change in kinetic energy is equal to the work done:

Using chain rule produces:

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}$$

is usually given in terms of x, not the substituted variable. For definite integrals, the answer is always a number.

Example

Evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$

Using $x = asin\theta$

$$dx = acos\theta d\theta$$

Note: dx, d(theta), and d(whatever) are not the same! You need to make a substitution here as well not only in "main body" of the integral.

The integral can be rewritten as:

$$\int \frac{acos\theta\ d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$$

And since $a^2 = a^2(\sin^2\theta + \cos^2\theta)$**:**

$$\int \frac{acos\theta\ d\theta}{\sqrt{a^2\cos^2\theta}} = \int 1d\theta = \theta + c = \arcsin\frac{x}{a} + c$$

Example

Evaluate $\int \sin^3 x \cos^5 x\, dx$

$$u = cosx => du = -sinxdx$$

$$\int sin\,x\,(1-\cos^2 x)\cos^5 x\,dx = \int (u^2-1)u^5du = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + c$$

More standard integrals

$f(x)$	$F(x)$	**Substitution**
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) + c$	$x = asin\theta$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arccosh}\left(\frac{x}{a}\right) + c$	$x = acosh\theta$
$\frac{1}{\sqrt{x^2+a^2}}$	$\text{arcsinh}\left(\frac{x}{a}\right) + c$	$x = asinh\theta$
$\frac{1}{x^2+a^2}$	$\arctan\left(\frac{x}{a}\right) + c$	$x = atan\theta$

4. $\int_2^{\infty} \frac{1}{x} dx = \lim_{\varepsilon \to \infty} [\ln x]_2^{\varepsilon}$

   The integral diverges since $\ln(\infty) = \infty$

5. $\int_{-1}^{1} \frac{1}{x} dx = \lim_{\varepsilon \to 0} [\ln|x|]_{-1}^{\varepsilon} + \lim_{\delta \to 0} [\ln|x|]_{\delta}^{1} = \infty - \infty$

   Which is undefined.

   Note: it is tempting to say it is 0, since the function is odd, but singularities basically make it a lot more complicated and it is not equal 0!

6. $\int_0^{\infty} \sin x \, dx = 1 - \lim_{x \to \infty} \cos x$

   Again it is undefined, because the function is periodical.

7. $\int \frac{\sin(lnx)}{x} dx = -\cos \ln x + c$

   Substitution $u = \ln x$ **was used.**

8. $\int_0^1 \frac{dx}{\sqrt{x}(1+x)} = 2[\arctan \sqrt{x}]_0^1 = \frac{\pi}{2}$

   Substitution $u = \sqrt{x}$ **was used.**

9. $\int \cos^3 3x \sin^2 3x \, dx = \frac{1}{3}\left(\frac{\sin^3 3x}{3} - \frac{\sin^5 3x}{5}\right) + c$

   Substitution $u = \sin 3x$ **was used.**

10. $\int_0^{\infty} v e^{\frac{-mv^2}{2kT}} dv = \frac{kT}{m}$

   Substitution $u = \frac{mv^2}{2kT}$ **was used.**

11. $\int \frac{dx}{3x^2-6x+7} = \frac{1}{2\sqrt{3}} \arctan\left(\frac{\sqrt{3}(x-1)}{2}\right) + c$

Exercise 6.2

More examples

1. $\int \frac{\sin^3 2x}{\sin^2 2x} dx = \frac{1}{2}(\sec 2x + \cos 2x) + c$

2. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}} dx = ln|e^x - e^{-x}| + c$

3. $\int \frac{x+3}{x^2-16} dx = \frac{1}{8}(ln|x+4| + 7ln|x-4|) + c$

4. $\int \frac{2x+5}{x^2+5} dx = ln|x^2+5| + \sqrt{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$

16. LINEAR ALGEBRA

Consider a set of linear equations:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

In linear algebra, we write those in terms of matrices and vectors.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

That equation is the same as the initial one. In more compact form:

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} e \\ f \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A\underline{x} = \underline{v}$$

That notation is useful since it just looks simpler and in cases with more than two equations, solving set of those is much more difficult than doing some calculations with matrices.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **is a 2x2 matrix (two rows, two columns). Some textbooks use $\underline{\underline{A}}$ notation for matrices, but it is not the standard one.**

Multiplication

Matrices (vector is also a matrix, it just has one column) can be multiplied only if the number of columns of the first one is the same as number of rows of the second one. $C = AB$ exists only if A is nxm and B is mxp. The size on resultant matrix will be nxp. Multiplication rule:

$$C = AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$$C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$$

where A_{ij} is the element of matrix A in row i and column j.

Note: in Einstein's summation convention the formula would look like that $V_i = A_{ij}X_j$ the sum in dropped since j is repeated only on the right side of equation which implies that we're adding, that notation is rarely used though.

Example

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 3 & 1 \times 2 + 2 \times 4 \\ 2 \times 1 - 1 \times 3 & 2 \times 2 - 1 \times 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ -1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 2 & 1 \times 2 - 2 \times 1 \\ 3 \times 1 + 4 \times 2 & 3 \times 2 - 4 \times 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 11 & 2 \end{pmatrix}$$

Example

Let $C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ **and** $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$**. Evaluate** CD **and** DC**.**

CD **is not defined.**

$$DC = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

Matrix multiplication is:

1. **Associative** $A(BC) = (AB)C$
2. **Distributive** $A(B + C) = AB + AC$
3. **NOT commutative** $AB \neq BA$

For matrix and a vector, the formula becomes:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Multiplication by scalar

Multiplication by a scalar requires multiplying all elements by that scalar. That operation is commutative.

$$B = cA \quad => \quad cA_{ij} = B_{ij}, \quad \forall i, j$$

Example

$$2\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix}$$

Equality

Two matrices are equal if and only if all their elements are equal, e.g.

$$A = B \quad <=> \quad A_{ij} = B_{ij}, \quad \forall i, j$$

Addition

Adding matrices requires just adding all their elements, e.g.

$$C = A + B \quad => \quad C_{ij} = A_{ij} + B_{ij}, \quad \forall i, j$$

Example

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}$$

Matrices multiplied by vectors transform them into new vectors. The operation is called linear mapping or just transformation.

Identity transformation

Identity transformation leaves the vector unchanged:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \underline{x}$$

I **is called the identity matrix.**

Zero

Multiplying any vector by $\underline{\underline{0}}$ **matrix** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **will always give** $\underline{0}$**:**

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\underline{x} = \underline{0}$$

Stretching

To stretch a vector by factor k**:**

$$A\underline{x} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\underline{x} = \begin{pmatrix} kx \\ ky \end{pmatrix} = k\underline{x}$$

Analogous to multiplying by scalar k**.**

Reflection in X axis

$$S\underline{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underline{x} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

To reflect with respect to Y axis, signs have to be flipped.

Rotations

To rotate a vector through an angle θ **in the counterclockwise direction:**

$$R_{\rightarrow} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Properties of matrices

Matrix is symmetric when it is equal to its transpose (symmetric with respect to main diagonal), $A = A^T$, e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

Matrix is orthogonal when its transpose equals its inverse, $A^{-1} = A^T$, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The trace of a matrix is the sum of elements lying on the main diagonal.

For matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$:

$$Tr(B) = 1 + 1 + 1 = 3$$

Using matrices, the definition of vector product can be reformulated. For vectors $\underline{a}$ and $\underline{b}$:

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix} = i\left(a_j b_k - a_k b_j\right) - j(a_i b_k - a_k b_i) + k(a_i b_j - a_j b_i)$$

Homogenous equations

These are in the form of $A\underline{x} = \underline{0}$.

For these kind of equations, there is always a trivial solution $\underline{x} = \underline{0}$, but that one is usually not particularly interesting. The existence of other solutions depends on the determinant of matrix A.

if $\det(A) \neq 0$, $\underline{x} = \underline{0}$ is a unique solution,

if $\det(A) = 0$, there is an infinite number of non-trivial solutions.

Comparing the coefficients produces $a = -1, b = -3, c = -6$. Thus, the general solution is:

$$y(x) + y_p = A_1 e^{-3x} + A_2 e^{\frac{1}{2}x} - x^2 - 3x - 6$$

Example

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$.

First, the complementary function:

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

$$y(x) = Ce^x + De^{2x}$$

Now, the particular integral for a polynomial:

$$y_p = Axe^{2x}$$

$$\frac{dy_p}{dx} = Ae^{2x} + 2Axe^{2x}$$

$$\frac{d^2y_p}{dx^2} = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

Substituting in the original ODE:

$$Ae^{2x} = e^{2x} => A = 1$$

Thus, the general solution is:

$$y(x) + y_p = Ce^x + De^{2x} + xe^{2x}$$

Example

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 6x + \sin(x)$.

Complementary function is the same as the one before so:

$$y(x) = Ce^x + De^{2x}$$

Now, the particular integral for a polynomial:

$$y_p = Ax + B + E\cos(x) + F\sin(x)$$

Substituting into the ODE:

$$(E - 3F)\cos(x) + (F + 3E)\sin(x) - 3A + 2B + 2Ax = 6x + sinx$$