1.3 Matrix algebra

1.3.1 Matrix equality

Two matrices are **equal** if they have the same size and if their corresponding elements are identical , i.e.

$$\mathbf{A} = \mathbf{B}$$

if and only if

$$a_{ij} = b_{ij} \quad \text{for } i = 1, ..., m; \ j = 1, ..., n$$

1.3.2 Matrix addition

Two matrices can only be added if they have the same size.

The result is another matrix of the same size.

We add matrices by adding their corresponding elements, i.e.

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$

is obtained (element-wise) via

$$a_{ij} = b_{ij} + c_{ij} \quad \text{for } i = 1, ..., m; \ j = 1, ..., n$$

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 10 & 1 & 23 \\ 5 & 16 & 3 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1+10 & 2+1 & 3+23 \\ 5+5 & 6+16 & 7+3 \end{pmatrix} = \begin{pmatrix} 11 & 3 & 26 \\ 10 & 22 & 10 \end{pmatrix}$$

Example

$$m_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

is obtained by suppressing the elements in row 1 and column 2 of matrix $\mathbf{A}$.

Cofactors

The **cofactor** c_{ij} is defined as the coefficient of a_{ij} in the determinant $\mathbf{A}$. If is given by the formula

$$c_{ij} = (-1)^{i+j} m_{ij}$$

where the **minor** is the determinant of order $(n-1) \times (n-1)$ formed by deleting the column and row containing a_{ij}.

Examples

$$c_{11} = (-1)^{1+1} m_{11} = +1 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$c_{23} = (-1)^{2+3} m_{23} = -1 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{11}a_{32} + a_{31}a_{12}$$

The matrix inverse can be computed as follows

1. Find the determinant $\det \mathbf{A}$
2. Find the cofactors of all elements in $\mathbf{A}$ and form a new matrix $\mathbf{C}$ of cofactors, where each element is replaced by its cofactor.
3. The inverse of $\mathbf{A}$ is now given as

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}}$$

Note: the inverse $\mathbf{A}^{-1}$ exists if (and only if) $\det \mathbf{A} \neq 0$.

Example Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}.$$

$$\begin{aligned} \det \mathbf{A} &= 1 \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} -3 & 1 \\ 3 & -2 \end{vmatrix} \\ &= 1 \times 3 + 1 \times (-3) + 2 \times 3 \\ &= 6 \end{aligned}$$

Since the determinant is nonzero an inverse exists.