26. A tree has been transplanted and after x years is growing at the rate of $1 + \frac{1}{(x+1)^2}$ meters per year. After two years it has reached a height of five meters. How tall was it when it was transplanted?

27. It is projected that t years from now the population of a certain country will be changing at the rate of $e^{0.02t}$ million per year. If the current population is 50 million, what will the population be 10 years from now?

Results.

1. $\frac{1}{5}e^{5x} + C$
2. $\frac{1}{6}(4x-1)\sqrt{4x-1} + C$
3. $\frac{1}{3}\ln|3x+5| + C$
4. $-e^{1-x} + C$
5. $e^{x^2-1} + C$
6. $\frac{1}{12}(x^2+1)^6 + C$
7. $(x^2+8)\sqrt{x^2+8} + C$
8. $\frac{4}{21}(x^3+1)^{\frac{7}{4}} + C$
9. $-\frac{1}{3(x^3+5)} + C$
10. $\frac{1}{26}(x^2+2x+5)^{13} + C$
11. $e^{x^3-x} + C$
12. $\frac{3}{5}\ln|x^5+5x^4+10x+12| + C$
13. $-\frac{3}{2(x^2-2x+6)} + C$
14. $\frac{3}{2}\ln|2x-1| + C$
15. $\frac{1}{2}\ln^2 5x + C$
16. $-\frac{1}{\ln x} + C$
17. $\frac{1}{2}\ln^2(x^2+1) + C$
18. $2e^{\sqrt{x}} + C$
19. $x + \ln|x-1| + C$
20. $\frac{2}{5}(x+1)^2\sqrt{x+1} - \frac{2}{3}(x+1)\sqrt{x+1} + C$
21. $-\frac{1}{3(x-5)^3} - \frac{1}{4(x-5)^4} + C$
22. $-\frac{7}{x-4} + \ln|x-4| + C$
23. $\frac{2x+1}{4}x - \frac{1}{4}\ln|2x+1| + C$

24. $f(x) = \frac{1}{3}(x^2+5)\sqrt{x^2+5} + 1$

25. $f(x) = -\frac{1}{3}\ln|1-3x^2| + 5$

26. $\frac{7}{3}$ meters

27. 61 million

Substituting $u = x^2 - 1$ and $\frac{1}{2}du = xdx$, you get

$$G(x) = \int x(x^2-1)^{10}dx = \frac{1}{2}\int u^{10}du = \frac{1}{22}u^{11} = \frac{1}{22}(x^2-1)^{11}.$$

Then

$$\begin{aligned}\int x^3(x^2-1)^{10}dx &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{22}\int 2x(x^2-1)^{11}dx = \\ &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{22}\frac{1}{12}(x^2-1)^{12} + C = \\ &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{264}(x^2-1)^{12} + C.\end{aligned}$$

(a) Use integration by parts to derive the formula

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$$

(b) Use the formula in part (a) to find $\int x^3 e^{5x}dx$.

Solution. (a) Since the factor e^{ax} is easy to integrate and the factor x^n is simplified by differentiation, try integration by parts with

$$g(x) = e^{ax} \qquad \text{and} \qquad f(x) = x^n.$$

Then,

$$G(x) = \int e^{ax}dx = \frac{1}{a}e^{ax} \qquad \text{and} \qquad f'(x) = nx^{n-1}$$

and so

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$$

(b) Apply the formula in part (a) with $a = 5$ and $n = 3$ to get

$$\int x^3 e^{5x}dx = \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\int x^2 e^{5x}dx.$$

Again, apply the formula in part (a) with $a = 5$ and $n = 2$ to find the new integral

$$\int x^2 e^{5x}dx = \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\int x e^{5x}dx.$$

Results.

1. $-(x+1)e^{-x}+C$
2. $(2x-4)e^{\frac{1}{2}x}+C$
3. $-5(x+5)e^{-\frac{1}{5}x}+C$
4. $(2-x)e^{x}+C$
5. $\frac{1}{2}x^2(\ln 2x-\frac{1}{2})+C$
6. $\frac{2}{3}x\,(x-6)^{\frac{3}{2}}-\frac{4}{15}\,(x-6)^{\frac{5}{2}}+C$
7. $\frac{1}{9}x(x+1)^9-\frac{1}{90}(x+1)^{10}+C$
8. $2x\,(x+2)^{\frac{1}{2}}-\frac{4}{3}\,(x+2)^{\frac{3}{2}}+C$
9. $x\,(2x+1)^{\frac{1}{2}}-\frac{1}{3}\,(2x+1)^{\frac{3}{2}}+C$
10. $-(x^2+2x+2)e^{-x}+C$
11. $\frac{1}{3}\left(x^2-\frac{2}{3}x+\frac{2}{9}\right)e^{3x}+C$
12. $(x^3-3x^2+6x-6)e^{x}+C$
13. $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$
14. $\frac{1}{2}x^2\left(\ln^2 x-\ln x+\frac{1}{2}\right)+C$
15. $-\frac{1}{x}(\ln x+1)+C$
16. $\frac{1}{36}x^4(x^4+5)^9-\frac{1}{360}(x^4+5)^{10}+C$

17. $f(x)=-(x+2)e^{-x}+\frac{3}{e}+5$

18. $f(x)=\frac{1}{4}x^2\left(\ln x-\frac{1}{2}\right)-\frac{5}{2}-\ln 2$

19.

   1. $s(t)=-2(t+2)e^{-\frac{t}{2}}+4$

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(a) Show that $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

(b) Use the formula in part (a) to evaluate $\int_{-1}^1 |x|\,dx$.

(c) Evaluate $\int_0^4 (1+|x-3|)^2 dx$.

Solution. (a) By the Newton-Leibniz formula, you have

$$\begin{aligned}\int_a^b f(x)dx + \int_b^c f(x)dx &= F(b) - F(a) + F(c) - F(b) = \\ &= F(c) - F(a) = \int_a^c f(x)dx.\end{aligned}$$

(b) Since $|x| = -x$ for $x \le 0$ and $|x| = x$ for $x \ge 0$, you have to break the given integral into two integrals

$$\int_{-1}^0 |x|\,dx = \int_{-1}^0 (-x)dx = -\frac{1}{2}x^2\Big|_{-1}^0 = 0 + \frac{1}{2} = \frac{1}{2}$$

and

$$\int_0^1 |x|\,dx = \int_0^1 xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Thus,

$$\int_{-1}^1 |x|\,dx = \int_{-1}^0 |x|\,dx + \int_0^1 |x|\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) Since $|x-3| = -x+3$ for $x \le 3$ and $|x-3| = x-3$ for $x \ge 3$, you get

$$\begin{aligned}\int_0^4 (1+|x-3|)^2 dx &= \int_0^3 [1+(-x+3)]^2 dx + \int_3^4 [1+(x-3)]^2 dx = \\ &= \int_0^3 (-x+4)^2 dx + \int_3^4 (x-2)^2 dx = \\ &= -\frac{1}{3}(-x+4)^3\Big|_0^3 + \frac{1}{3}(x-2)^3\Big|_3^4 = \\ &= -\frac{1}{3} + \frac{64}{3} + \frac{8}{3} - \frac{1}{3} = \frac{70}{3}.\end{aligned}$$

9. (a) Show that if F is an antiderivative of f, then

$$\int_a^b f(-x)dx = -F(-b) + F(-a)$$

6.6 Area and Integration

In problems 1 through 9 find the area of the region R.

1. R is the triangle with vertices $(-4,0)$, $(2,0)$ and $(2,6)$.
   Solution. From the corresponding graph (Figure 6.1) you see that the region in question is bellow the line $y = x + 4$ above the x axis, and extends from $x = -4$ to $x = 2$.

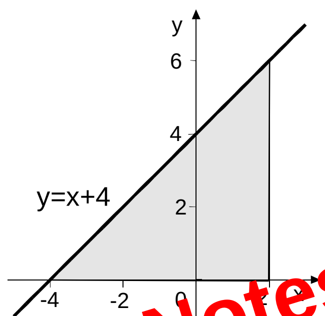

Figure 6.1.

Hence,

$$A = \int_{-4}^{2} (x+4)dx = \left(\frac{1}{2}x^2 + 4x\right)\Big|_{-4}^{2} = (2+8) - (8-16) = 18.$$

2. R is the region bounded by the curve $y = e^x$, the lines $x = 0$ and $x = \ln\frac{1}{2}$, and the x axis.
   Solution. Since $\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2 \simeq -0.7$, from the corresponding graph (Figure 6.2) you see that the region in question is bellow the line $y = e^x$ above the x axis, and extends from $x = \ln\frac{1}{2}$ to $x = 0$.

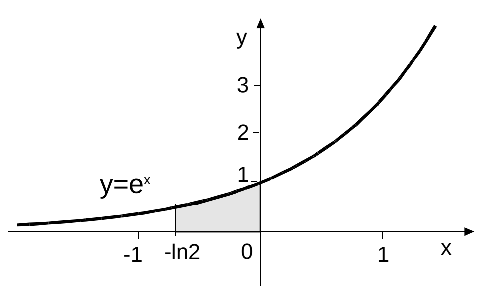

Figure 6.2.