Now we try to integrate along another curve $C_2 : \mathbf{r}(t) = (t, t, t)$. So $\mathbf{r}'(t) = (1, 1, 1)$.

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int \mathbf{F} \cdot \mathbf{r}'(t) \mathrm{d}t \\ &= \int_0^1 te^t + 2t^2 \, \mathrm{d}t \\ &= \frac{5}{3}.\end{aligned}$$

We see that the line integral depends on the curve C in general, not just $\mathbf{a}, \mathbf{b}$.

We can also use the arclength s as the parameter. Since $\mathrm{d}\mathbf{r} = \mathbf{t} \, \mathrm{d}s$, with $\mathbf{t}$ being the unit tangent vector, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t} \, \mathrm{d}s.$$

Note that we do not necessarily have to integrate $\mathbf{F} \cdot \mathbf{t}$ with respect to s. We can also integrate a scalar function as a function of s, $\int_C f(s) \, \mathrm{d}s$. By convention, this is calculated in the direction of increasing s. In particular, we have

$$\int_C 1 \, \mathrm{d}s = \text{length of C}.$$

Definition (Closed curve)**.** A *closed curve* is a curve with the same start and end point. The line integral along a closed curve is (sometimes) written as $\oint$ and is (sometimes) called the *circulation* of $\mathbf{F}$ around C.

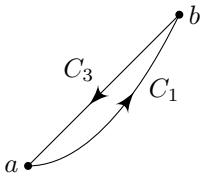
Sometimes we are not that lucky and our curve is not smooth. For example, the graph of an absolute value function is not smooth. However, often we can break it apart into many smaller segments, each of which is smooth. Alternatively, we can write the curve as a sum of smooth curves. We call these *piecewise smooth* curves.

Definition (Piecewise smooth curve)**.** A *piecewise smooth curve* is a curve $C = C_1 + C_2 + \cdots + C_n$ with all C_i smooth with regular parametrisations. The line integral over a piecewise smooth C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Example. Take the example above, and let $C_3 = -C_2$. Then $C = C_1 + C_3$ is piecewise smooth but not smooth. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \left(\frac{e}{2} + \frac{1}{4}\right) - \frac{5}{3} \\ &= -\frac{17}{12} + \frac{e}{2}.\end{aligned}$$



2.3 Gradients and Differentials

Recall that the line integral depends on the actual curve taken, and not just the end points. However, for some nice functions, the integral *does* depend on the end points only.

Theorem. If $\mathbf{F} = \nabla f(\mathbf{r})$, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

where $\mathbf{b}$ and $\mathbf{a}$ are the end points of the curve.

In particular, the line integral does *not* depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when C is a closed curve, then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

Proof. Let $\mathbf{r}(u)$ be any parametrization of the curve, and suppose $\mathbf{a} = \mathbf{r}(\alpha)$, $\mathbf{b} = \mathbf{r}(\beta)$. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = \int \nabla f \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\,\mathrm{d}u.$$

So by the chain rule, this is equal to

$$\int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}u} f(\mathbf{r}(u))\,\mathrm{d}u = [f(\mathbf{r}(u))]_\alpha^\beta = f(\mathbf{b}) - f(\mathbf{a}). \qquad \square$$

Definition (Conservative vector field)**.** If $\mathbf{F} = \nabla f$ for some f, the $\mathbf{F}$ is called a *conservative vector field*.

The name *conservative* comes from mechanics, where conservative vector fields represent conservative forces that conserve energy. This is since if the force is conservative, then the integral (i.e. work done) about a closed curve is 0, which means that we cannot gain energy after travelling around the loop.

It is convenient to treat differentials $\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_i \mathrm{d}x_i$ as if they were objects by themselves, which we can integrate along curves if we feel like doing so.

Then we can define

Definition (Exact differential)**.** A differential $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *exact* if there is an f such that $\mathbf{F} = \nabla f$. Then

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

To test if this holds, we can use the necessary condition

Note that in polar coordinates, we are integrating over a rectangle and the function is separable. So this is equal to

$$\begin{aligned} &= \left[-e^{-\rho^2/2}\right]_0^R [\varphi]_0^{\pi/2} \\ &= \frac{\pi}{2}\left(1 - e^{-R^2/2}\right). \end{aligned} \tag{$*$}$$

Note that the integral exists as $R \to \infty$.

Now we take the case of $x, y \to \infty$ and consider the original integral.

$$\begin{aligned} \int_D f \, \mathrm{d}A &= \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)/2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \left(\int_0^{\infty} e^{-x^2/2} \, \mathrm{d}x\right)\left(\int_0^{\infty} e^{-y^2/2} \, \mathrm{d}y\right) \\ &= \frac{\pi}{2} \end{aligned}$$

where the last line is from (*). So each of the two integrals must be $\sqrt{\pi/2}$, i.e.

$$\int_0^{\infty} e^{-x^2/2} \, \mathrm{d}x = \sqrt{\frac{\pi}{2}}.$$

3.3 Generalization to $\mathbb{R}^3$

We will do exactly the same thing as we just did, but with one more dimension:

Definition (Volume integral)**.** Consider a volume $V \subseteq \mathbb{R}^3$ with position vector $\mathbf{r} = (x, y, z)$. We approximate V by N small disjoint subsets of some simple shape (e.g. cuboids) labelled by I, volume δV_I, contained within a solid sphere of diameter ℓ.

Assume that as $\ell \to 0$ and $N \to \infty$, the union of the small subsets tend to V. Then

$$\int_V f(\mathbf{r}) \, \mathrm{d}V = \lim_{\ell \to 0} \sum_I f(\mathbf{r}_I^*)\delta V_I,$$

where $\mathbf{r}_I^*$ is any chosen point in each small subset.

To evaluate this, we can take $\delta V_I = \delta x \delta y \delta z$, and take $\delta x \to 0$, $\delta y \to 0$ and δz in some order. For example,

$$\int_V f(\mathbf{r}) \, \mathrm{d}v = \int_D \left(\int_{Z_{xy}} f(x, y, z) \, \mathrm{d}z\right) \mathrm{d}x \, \mathrm{d}y.$$

So we integrate $f(x, y, z)$ over z at each point (x, y), then take the integral of that over the area containing all required (x, y).

Alternatively, we can take the area integral first, and have

$$\int_V f(\mathbf{r}) \, \mathrm{d}V = \int_z \left(\int_{D_z} f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y\right) \mathrm{d}z.$$

Again, if we take $f = 1$, then we obtain the volume of V.

Often, $f(\mathbf{r})$ is the density of some quantity, and is usually denoted by ρ. For example, we might have mass density, charge density, or probability density. $\rho(\mathbf{r})\delta V$ is then the amount of quantity in a small volume δV at $\mathbf{r}$. Then $\int_V \rho(\mathbf{r}) \, \mathrm{d}V$ is the total amount of quantity in V.

4 Surfaces and surface integrals

4.1 Surfaces and Normal

So far, we have learnt how to do calculus with regions of the plane or space. What we would like to do now is to study surfaces in $\mathbb{R}^3$. The first thing to figure out is how to specify surfaces. One way to specify a surface is to use an equation. We let f be a smooth function on $\mathbb{R}^3$, and c be a constant. Then $f(\mathbf{r}) = c$ defines a smooth surface (e.g. $x^2 + y^2 + z^2 = 1$ denotes the unit sphere).

Now consider any curve $\mathbf{r}(u)$ on S. Then by the chain rule, if we differentiate $f(\mathbf{r}) = c$ with respect to u, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u}[f(\mathbf{r}(u))] = \nabla f \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = 0.$$

This means that ∇f is always perpendicular to $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}$. Since $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}$ is the tangent to the curve, ∇f is perpendicular to the tangent. Since this is true for *any* curve $\mathbf{r}(u)$, ∇f is perpendicular to any tangent of the surface. Therefore

Proposition. ∇f is the normal to the surface $f(\mathbf{r}) = c$.

Example.

(i) Take the sphere $f(\mathbf{r}) = x^2 + y^2 + z^2 = c$ for $c > 0$. Then $\nabla f = 2(x, y, z) = 2\mathbf{r}$, which is clearly normal to the sphere.

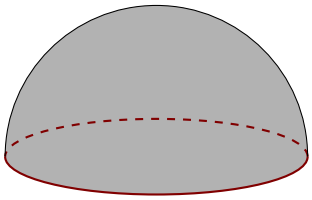
(ii) Take $f(\mathbf{r}) = x^2 + y^2 - z^2 = c$, which is a hyperboloid. Then $\nabla f = 2(x, y, -z)$.

In the special case when $c = 0$, we have a double cone with a singular apex $\mathbf{0}$. Here $\nabla f = \mathbf{0}$, and we cannot find a meaningful direction of normal.

Definition (Boundary)**.** A surface S can be defined to have a *boundary* ∂S consisting of a piecewise smooth curve. If we define S as in the above examples but with the additional restriction $z \geq 0$, then ∂S is the circle $x^2 + y^2 = c$, $z = 0$.

A surface is *bounded* if it can be contained in a solid sphere, *unbounded* otherwise. A bounded surface with no boundary is called *closed* (e.g. sphere).

Example.



The boundary of a hemisphere is a circle (drawn in red).

Definition (Orientable surface)**.** At each point, there is a unit normal $\mathbf{n}$ that's unique up to a sign.

If we can find a consistent choice of $\mathbf{n}$ that varies smoothly across S, then we say S is *orientable*, and the choice of sign of $\mathbf{n}$ is called the *orientation* of the surface.

The parametrizations we use will all be regular.

Given a surface, how could we, say, find its area? We can use our parametrization. Suppose points on the surface are given by $\mathbf{r}(u, v)$ for $(u, v) \in D$. If we want to find the area of D itself, we would simply integrate

$$\int_D \mathrm{d}u\, \mathrm{d}v.$$

However, we are just using u and v as arbitrary labels for points in the surface, and one unit of area in D does not correspond to one unit of area in S. Instead, suppose we produce a small rectangle in D by changing u and v by small $\delta u, \delta v$. In D, this corresponds to a rectangle with vertices $(u, v), (u + \delta u, v), (u, v + \delta v), (u + \delta u, v + \delta v)$, and spans an area $\delta u \delta v$. In the surface S, these small changes $\delta u, \delta v$ correspond to changes $\frac{\partial \mathbf{r}}{\partial u}\delta u$ and $\frac{\partial \mathbf{r}}{\partial v}\delta v$, and these span a *vector* area of

$$\delta \mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \delta u \delta v = \mathbf{n}\, \delta S.$$

Note that the order of u, v gives the choice of the sign of the unit normal.

The actual area is then given by

$$\delta S = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \delta u\, \delta v.$$

Making these into differentials instead of deltas, we have

Proposition. The *vector area element* is

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \mathrm{d}u\, \mathrm{d}v.$$

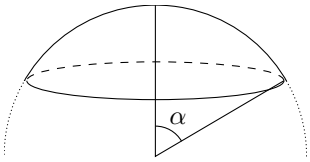
The *scalar area element* is

$$\mathrm{d}S = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

By summing and taking limits, the area of S is

$$\int_S \mathrm{d}S = \int_D \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

Example. Consider again the part of the sphere of radius a with $0 \leq \theta \leq \alpha$.



Then we have

$$\mathbf{r}(\theta, \varphi) = (a \cos\varphi \sin\theta, a \sin\theta \sin\varphi, a\cos\theta) = a\mathbf{e}_r.$$

So we find

$$\frac{\partial \mathbf{r}}{\partial \theta} = a\mathbf{e}_\theta.$$

5 Geometry of curves and surfaces

Let $\mathbf{r}(s)$ be a curve parametrized by arclength s. Since $\mathbf{t}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$ is a unit vector, $\mathbf{t} \cdot \mathbf{t} = 1$. Differentiating yields $\mathbf{t} \cdot \mathbf{t}' = 0$. So $\mathbf{t}'$ is a normal to the curve if $\mathbf{t}' \neq 0$.

We define the following:

Definition (Principal normal and curvature)**.** Write $\mathbf{t}' = \kappa \mathbf{n}$, where $\mathbf{n}$ is a unit vector and $\kappa > 0$. Then $\mathbf{n}(s)$ is called the *principal normal* and $\kappa(s)$ is called the *curvature*.

Note that we *must* be differentiating against s, not any other parametrization! If the curve is given in another parametrization, we can either change the parametrization or use the chain rule.

We take a curve that can Taylor expanded around $s = 0$. Then

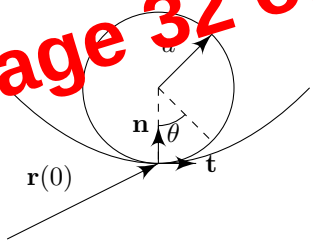
$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + O(s^3).$$

We know that $\mathbf{r}' = \mathbf{t}$ and $\mathbf{r}'' = \mathbf{t}'$. So we have

$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{t}(0) + \frac{1}{2}\kappa(0)s^2\mathbf{n} + O(s^3).$$

How can we interpret κ as the curvature? Suppose we want to approximate the curve near $\mathbf{r}(0)$ by a circle. We would expect a more "curved" curve would be approximated by a circle of smaller radius. So κ should be inversely proportional to the radius of the circle. In fact, we will show that $\kappa = 1/a$, where a is the radius of the best-fit circle.

Consider the vector equation for a circle passing through $\mathbf{r}(0)$ with radius a in the plane defined by $\mathbf{t}$ and $\mathbf{n}$.



Then the equation of the circle is

$$\mathbf{r} = \mathbf{r}(0) + a(1 - \cos\theta)\mathbf{n} + a\sin\theta\mathbf{t}.$$

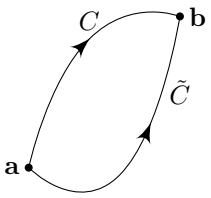
We can expand this to obtain

$$\mathbf{r} = \mathbf{r}(0) + a\theta\mathbf{t} + \frac{1}{2}\theta^2 a\mathbf{n} + o(\theta^3).$$

Since the arclength $s = a\theta$, we obtain

$$\mathbf{r} = \mathbf{r}(0) + s\mathbf{t} + \frac{1}{2}\frac{1}{a}s^2\mathbf{n} + O(s^3).$$

As promised, $\kappa = 1/a$, for a the radius of the circle of best fit.



If S is any surface with boundary $\partial S = C - \tilde{C}$, By Stokes' theorem,

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

But $\nabla \times \mathbf{F} = 0$. So

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0,$$

or

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\tilde{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \qquad \square$$

Proposition. If (ii) $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of C for fixed end points and orientation, then (i) $\mathbf{F} = \nabla f$ for some scalar field f.

Proof. We fix $\mathbf{a}$ and define $f(\mathbf{r}) = \int_C \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}'$ for any curve from $\mathbf{a}$ to $\mathbf{r}$. Assuming (ii), f is well-defined. For small changes $\mathbf{r}$ to $\mathbf{r} + \delta\mathbf{r}$, then $C + \delta C$ is a small extension of C by δC. Then

$$\begin{aligned} f(\mathbf{r} + \delta\mathbf{r}) &= \int_{C+\delta C} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' \\ &= \int_C \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' + \int_{\delta C} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' \\ &= f(\mathbf{r}) + \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} + o(\delta\mathbf{r}). \end{aligned}$$

So

$$\delta f = f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} + o(\delta\mathbf{r}).$$

But the definition of grad is exactly

$$\delta f = \nabla f \cdot \delta\mathbf{r} + o(\delta\mathbf{r}).$$

So we have $\mathbf{F} = \nabla f$. $\square$

Note that these results assume $\mathbf{F}$ is defined on the whole of $\mathbb{R}^3$. It also works of $\mathbf{F}$ is defined on a *simply connected* domain D, ie a subspace of $\mathbb{R}^3$ without holes. By definition, this means that any two curves $C, \tilde{C}$ with fixed end points can be smoothly deformed into one another (alternatively, any loop can be shrunk into a point).

If we have a smooth transformation from C to $\tilde{C}$, the process sweeps out a surface bounded by C and $\tilde{C}$. This is required by the proof that (iii) $\Rightarrow$ (ii).

If D is not simply connected, then we obtain a multi-valued $f(\mathbf{r})$ on D in general (for the proof (ii) $\Rightarrow$ (i)). However, we can choose to restrict to a subset $D_0 \subseteq D$ such that $f(\mathbf{r})$ is single-valued on D_0.

9 Orthogonal curvilinear coordinates

9.1 Line, area and volume elements

In this chapter, we study funny coordinate systems. A coordinate system is, roughly speaking, a way to specify a point in space by a set of (usually 3) numbers. We can think of this as a function $\mathbf{r}(u, v, w)$.

By the chain rule, we have

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w}\mathrm{d}w$$

For a good parametrization,

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \neq 0,$$

i.e. $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}$ and $\frac{\partial \mathbf{r}}{\partial w}$ are linearly independent. These vectors are tangent to the curves parametrized by u, v, w respectively when the other two are being fixed.

Even better, they should be orthogonal:

Definition (Orthogonal curvilinear coordinates)**.** u, v, w are *orthogonal curvilinear* if the tangent vectors are orthogonal.

We can then set

$$\frac{\partial \mathbf{r}}{\partial u} = h_u \mathbf{e}_u, \quad \frac{\partial \mathbf{r}}{\partial v} = h_v \mathbf{e}_v, \quad \frac{\partial \mathbf{r}}{\partial w} = h_w \mathbf{e}_w$$

with $h_u, h_v, h_w > 0$ and $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$ form an *orthonormal* right-handed basis (i.e. $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$). Then

$$\mathrm{d}\mathbf{r} = h_u \mathbf{e}_u \, \mathrm{d}u + h_v \mathbf{e}_v \, \mathrm{d}v + h_w \mathbf{e}_w \, \mathrm{d}w,$$

and h_u, h_v, h_w determine the changes in length along each orthogonal direction resulting from changes in u, v, w. Note that clearly by definition, we have

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|.$$

Example.

(i) In cartesian coordinates, $\mathbf{r}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Then $h_x = h_y = h_z = 1$, and $\mathbf{e}_x = \hat{\mathbf{i}}, \mathbf{e}_y = \hat{\mathbf{j}}$ and $\mathbf{e}_z = \hat{\mathbf{k}}$.

(ii) In cylindrical polars, $\mathbf{r}(\rho, \varphi, z) = \rho[\cos\varphi\hat{\mathbf{i}} + \sin\varphi\hat{\mathbf{j}}] + z\hat{\mathbf{k}}$. Then $h_\rho = h_z = 1$, and

$$h_\varphi = \left|\frac{\partial \mathbf{r}}{\partial \varphi}\right| = |(-\rho\sin\varphi, \rho\sin\varphi, 0)| = \rho.$$

The basis vectors $\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{e}_z$ are as in section 1.

(iii) In spherical polars,

$$\mathbf{r}(r, \theta, \varphi) = r(\cos\varphi\sin\theta\hat{\mathbf{i}} + \sin\theta\sin\varphi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}).$$

Then $h_r = 1, h_\theta = r$ and $h_\varphi = r\sin\theta$.

The condition $\int_C \mathbf{g} \cdot d\mathbf{r} = 0$ for any closed C can be re-written by Stoke's theorem as

$$\int_S \nabla \times \mathbf{g} \cdot d\mathbf{S} = 0,$$

where S is bounded by the closed curve C. This is true for arbitrary S. So

$$\nabla \times \mathbf{g} = 0.$$

In our example above, $\nabla \times \mathbf{g} = 0$ due to spherical symmetry. But here we showed that it is true for all cases.

Note that we exploited symmetry to solve Gauss' law. However, if the mass distribution is not sufficiently symmetrical, Gauss' law in integral form can be difficult to use. But we can rewrite it in differential form. Suppose

$$M = \int_V \rho(\mathbf{r})\, dV,$$

where ρ is the mass density. Then by Gauss' theorem

$$\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM \Rightarrow \int_V \nabla \cdot \mathbf{g}\, dV = \int_V -4\pi G\rho\, dV.$$

Since this is true for all V, we must have

Law (Gauss' Law for gravitation in differential form)**.**

$$\nabla \cdot \mathbf{g} = -4\pi G\rho.$$

Since $\nabla \times \mathbf{g} = 0$, we can introduce a gravitational potential $\varphi(\mathbf{r})$ with $\mathbf{g} = -\nabla\varphi$. Then Gauss' Law becomes

$$\nabla^2 \varphi = 4\pi G\rho.$$

In the example with spherical symmetry, we can solve that

$$\varphi(r) = -\frac{GM}{r}$$

for $r > a$.

10.2 Laws of electrostatics

Consider a distribution of electric charge at rest. They produce a force on a charge q, at rest at $\mathbf{r}$, which is proportional to q.

Definition (Electric field)**.** The force produced by electric charges on another charge q is $\mathbf{F} = q\mathbf{E}(\mathbf{r})$, where $\mathbf{E}(\mathbf{r})$ is the *electric field*, or force per unit charge.

Again, this is conservative. So

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 0$$

for any closed curve C. It also obeys

In words, this says that the value at the center of a sphere is the average of the values on the surface on the sphere.

Proof. Note that $\bar{\varphi}(R) \to \varphi(\mathbf{a})$ as $R \to 0$. We take spherical coordinates (u, θ, χ) centered on $\mathbf{r} = \mathbf{a}$. The scalar element (when $u = R$) on S_R is

$$\mathrm{d}S = R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\chi.$$

So $\frac{\mathrm{d}S}{R^2}$ is independent of R. Write

$$\bar{\varphi}(R) = \frac{1}{4\pi} \int \varphi \frac{\mathrm{d}S}{R^2}.$$

Differentiate this with respect to R, noting that $\mathrm{d}S/R^2$ is independent of R. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\bar{\varphi}(R) = \frac{1}{4\pi R^2} \int \left.\frac{\partial \varphi}{\partial u}\right|_{u=R} \mathrm{d}S$$

But

$$\frac{\partial \varphi}{\partial u} = \mathbf{e}_u \cdot \nabla \varphi = \mathbf{n} \cdot \nabla \varphi = \frac{\partial \varphi}{\partial \mathbf{n}}$$

on S_R. So

$$\frac{\mathrm{d}}{\mathrm{d}R}\bar{\varphi}(R) = \frac{1}{4\pi R^2} \int_{S_R} \nabla \varphi \cdot \mathrm{d}\mathbf{S} = \frac{1}{4\pi R^2} \int_{V_R} \nabla^2 \varphi \, \mathrm{d}V = 0$$

by divergence theorem. So $\bar{\varphi}(R)$ does not depend on R, and the result follows. □

11.2.2 The maximum (or minimum) principle

In this section, we will talk about maxima of functions. It should be clear that the results also hold for minima.

Definition (Local maximum). We say that $\varphi(\mathbf{r})$ has a *local maximum* at $\mathbf{a}$ if for some $\varepsilon > 0$, $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$ when $0 < |\mathbf{r} - \mathbf{a}| < \varepsilon$.

Proposition (Maximum principle)**.** If a function φ is harmonic on a region V, then φ cannot have a maximum at an interior point of $\mathbf{a}$ of V.

Proof. Suppose that φ had a local maximum at $\mathbf{a}$ in the interior. Then there is an ε such that for any $\mathbf{r}$ such that $0 < |\mathbf{r} - \mathbf{a}| < \varepsilon$, we have $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$.

Note that if there is an ε that works, then any smaller ε will work. Pick an ε sufficiently small such that the region $|\mathbf{r} - \mathbf{a}| < \varepsilon$ lies within V (possible since $\mathbf{a}$ lies in the interior of V).

Then for any $\mathbf{r}$ such that $|\mathbf{r} - \mathbf{a}| = \varepsilon$, we have $\varphi(\mathbf{r}) < \varphi(\mathbf{a})$.

$$\bar{\varphi}(\varepsilon) = \frac{1}{4\pi R^2} \int_{S_R} \varphi(\mathbf{r}) \, \mathrm{d}S < \varphi(\mathbf{a}),$$

which contradicts the mean value property. □

We can understand this by performing a local analysis of stationary points by differentiation. Suppose at $\mathbf{r} = \mathbf{a}$, we have $\nabla \varphi = 0$. Let the eigenvalues of the Hessian matrix $H_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ be λ_i. But since φ is harmonic, we have $\nabla^2 \varphi = 0$,

12 Maxwell's equations

12.1 Laws of electromagnetism

Maxwell's equations are a set of four equations that describe the behaviours of electromagnetism. Together with the Lorentz force law, these describe *all* we know about (classical) electromagnetism. All other results we know are simply mathematical consequences of these equations. It is thus important to understand the mathematical properties of these equations.

To begin with, there are two fields that govern electromagnetism, known as the *electric* and *magnetic* field. These are denoted by $\mathbf{E}(r,t)$ and $\mathbf{B}(r,t)$ respectively.

To understand electromagnetism, we need to understand how these fields are formed, and how these fields affect charged particles. The second is rather straightforward, and is given by the Lorentz force law.

Law (Lorentz force law)**.** A point charge q experiences a force of

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}).$$

The dynamics of the field itself is governed by *Maxwell's equations*. To state the equations, we need to introduce two more concepts.

Definition (Charge and current density)**.** $\rho(\mathbf{r},t)$ is the *charge density*, defined as the charge per unit volume.

$\mathbf{j}(\mathbf{r},t)$ is the *current density*, defined as the electric current per unit area of cross section.

Then Maxwell's equations say

Law (Maxwell's equations)**.**

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}\\
\nabla \cdot \mathbf{B} &= 0\\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0\\
\nabla \times \mathbf{B} - \mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j},
\end{aligned}$$

where ε_0 is the electric constant (permittivity of free space) and μ_0 is the magnetic constant (permeability of free space), which are constants determined experimentally.

We can quickly derive some properties we know from these four equations. The conservation of electric charge comes from taking the divergence of the last equation.

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0} - \mu_0\varepsilon_0 \frac{\partial}{\partial t} \underbrace{(\nabla \cdot \mathbf{E})}_{=\rho/\varepsilon_0} = \mu_0 \nabla \cdot \mathbf{j}.$$

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

We can also take the volume integral of the first equation to obtain

$$\int_V \nabla \cdot \mathbf{E} \, \mathrm{d}V = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V = \frac{Q}{\varepsilon_0}.$$

By the divergence theorem, we have

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

which is Gauss' law for electric fields

We can integrate the second equation to obtain

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

This roughly states that there are no "magnetic charges".

The remaining Maxwell's equations also have integral forms. For example,

$$\int_{C=\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = \int_S \nabla \times \mathbf{E} \, \mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S},$$

where the first equality is from from Stoke's theorem. This says that a changing magnetic field produces a current.

12.2 Static charges and steady currents

If $\rho, \mathbf{j}, \mathbf{E}, \mathbf{B}$ are all independent of time, $\mathbf{E}$ and $\mathbf{B}$ are no longer linked.

We can solve the equations for electric fields

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$$
$$\nabla \times \mathbf{E} = \mathbf{0}$$

Second equation gives $\mathbf{E} = -\nabla\varphi$. Substituting into first gives $\nabla^2\varphi = -\rho/\varepsilon_0$.

The equations for the magnetic field are

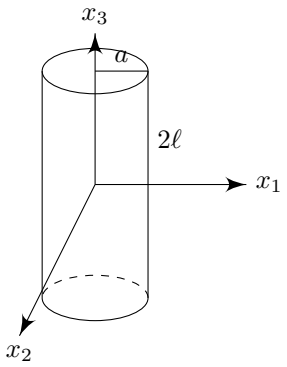
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

First equation gives $B = \nabla \times \mathbf{A}$ for some *vector potential* $\mathbf{A}$. But the vector potential is not well-defined. Making the transformation $\mathbf{A} \mapsto \mathbf{A} + \nabla\chi(\mathbf{x})$ produces the same $\mathbf{B}$, since $\nabla \times (\nabla\chi) = 0$. So choose χ such that $\nabla \cdot \mathbf{A} = 0$. Then

$$\nabla^2 \mathbf{A} = \nabla(\underbrace{\nabla \cdot \mathbf{A}}_{=0}) - \nabla \times (\underbrace{\nabla \times \mathbf{A}}_{\mathbf{B}}) = -\mu_0 \mathbf{j}.$$

In summary, we have

Electrostatics	Magnetostatics
$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ $\nabla \times \mathbf{E} = \mathbf{0}$ $\nabla^2\varphi = -\rho/\varepsilon_0$ ε_0 sets the scale of electrostatic effects, e.g. the Coulomb force	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{B} = \mu_0\mathbf{j}$ $\nabla^2\mathbf{A} = -\mu_0\mathbf{j}$. μ_0 sets the scale of magnetic effects, e.g. force between two wires with currents.



Use cylindrical polar coordinates:

$$\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \\ x_3 &= x_3 \\ \mathrm{d}V &= r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x_3 \end{aligned}$$

We have

$$\begin{aligned} I_{33} &= \int_V \rho_0(x_1^2 + x_2^2)\,\mathrm{d}V \\ &= \rho_0 \int_0^a \int_0^{2\pi} \int_{-\ell}^{\ell} r^2 (r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x_2) \\ &= \rho_0 \cdot 2\pi \cdot 2\ell \left[\frac{r^4}{4}\right]_0^a \\ &= \frac{\rho_0 \pi a^4 \ell}{2} \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_{11} &= \int_V \rho_0(x_2^2 + x_3^2)\,\mathrm{d}V \\ &= \rho_0 \int_0^a \int_0^{2\pi} \int_{-\ell}^{\ell} (r^2\sin^2\theta + x_3^2) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x_3 \\ &= \rho_0 \int_0^a \int_0^{2\pi} r\left(r^2\sin^2\theta\,[x_3]_{-\ell}^{\ell} + \left[\frac{x_3^3}{3}\right]_{-\ell}^{\ell}\right)\mathrm{d}\theta\,\mathrm{d}r \\ &= \rho_0 \int_0^a \int_0^{2\pi} r\left(r^2\sin^2\theta 2\ell + \frac{2}{3}\ell^3\right)\mathrm{d}\theta\,\mathrm{d}r \\ &= \rho_0\left(2\pi a\cdot\frac{2}{3}\ell^3 + 2\ell\int_0^a r^2\,\mathrm{d}r\int_0^{2\pi}\sin^2\theta\right) \\ &= \rho_0\pi a^2\ell\left(\frac{a^2}{2} + \frac{2}{3}\ell^2\right) \end{aligned}$$

By symmetry, the result for I_{22} is the same.