Note : LUB of a bounded above set may or may not be a member of the set. If it does, it is the maximum element of the set.

10. **Completeness Axiom:** *Every non-empty bounded above set of real numbers has its LUB in* $\mathbb{R}$.

    Example: Let $A = \{x \in \mathbb{Q} : x > 0 \wedge x^2 < 2\}$

    $\therefore\ x \in A \Rightarrow x > 0 \wedge x^2 < 2 \Rightarrow 0 < x < \sqrt{2}$. Thus A is a non-empty bounded above set of real numbers.

    By Completeness Axiom, $\sup A \in \mathbb{R}$. Let $\sup A = u$. We show that $u \notin \mathbb{Q}$.

    If possible let $u \in \mathbb{Q}$. Then $u > 0$ and $u \leq \sqrt{2}$ and $u \in \mathbb{Q} \Rightarrow 0 < u < \sqrt{2}$. [since $\sqrt{2}$ is irrational]

    Let $v = \dfrac{4+3u}{3+2u}$. Then $v > 0$ and $v \in \mathbb{Q}$

    $v - u = \dfrac{4+3u}{3+2u} - u = \dfrac{4-2u^2}{3+2u} = \dfrac{2(2-u^2)}{3+2u} > 0 \Rightarrow v > u$

    Also, $v^2 - 2 = \dfrac{16+24u+9u^2}{9+12u+4u^2} - 2 = \dfrac{u^2-2}{(3+2u)^2} < 0 \Rightarrow v^2 < 2$

    Thus $v \in A$ which contradicts the fact that $u = \sup A$. Hence $u \notin \mathbb{Q}$

    Remark: `The above example shows that the supremum of a set of rational numbers may not be rational. Thus Completeness Axiom\LUB Axiom does not hold for` $\mathbb{Q}$. `Therefore` $\mathbb{R}$ `is a complete ordered field and` $\mathbb{Q}$ `is an incomplete ordered field.`

11. **Archimedean Property of Real Numbers**

    There are two ways to state the Archimedean property

    (a) The set $\mathbb{N}$ of natural numbers is unbounded above.
    Proof: $\mathbb{N} \neq \emptyset, N \subset \mathbb{R}$.
    If possible, let $\mathbb{N}$ be bounded above. Thus by LUB Axiom, $\sup \mathbb{N} \in \mathbb{R}$. Let $\sup \mathbb{N} = \mu$ Then

      i. $n \leq \mu\ \ \forall n \in \mathbb{N}$
      ii. corresponding to $\varepsilon = 1$, $\exists p \in \mathbb{N}$ suc that $p > \mu - 1 \Rightarrow p + 1 > \mu$ and $p + 1 \in \mathbb{N}$

    This contradicts the fact that $\mu = \sup \mathbb{N}$. Hence $\mathbb{N}$ is unbounded above.

    (b) (EQUIVALENT STATEMENT): If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $nx > y$.
    Proof: If $y \leq 0$ then $n = 1$ serves the purpose.
    Let us assume that $y > 0$. If possible let $nx \leq y\ \ \forall n \in \mathbb{N}$.
    Then the set $S = \{nx : n \in \mathbb{N}\}$ is a non-empty bounded above subset of $\mathbb{R}$, y being an upper bound.
    By the LUB axiom, $\sup S$ exist in $\mathbb{R}$.
    Let $\sup S = u$. Then