$$x\frac{dy}{dx} + y = 2.$$

NB: $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$. Integrating both sides we get $\int [\frac{d}{dx}(xy)]\, dx = \int (x\frac{dy}{dx} + y)\, dx$

$$xy = \int (x\frac{dy}{dx} + y)\, dx$$

Integrating both sides

$$\int [x\frac{dy}{dx} + y]\, dx = \int 2\, dx$$

$$xy = 2x + c$$

$$y = \frac{2x + c}{x}.$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation by noticing, by the *Product Rule*, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + c$$

If we had been given the differential equation in the form $y' + (\frac{1}{x})y = \frac{2}{x}$, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides by a suitable function λ called an ***integrating factor***. We try to find λ so that the left side of the ODE, when multiplied by λ, becomes the derivative of the product λy:

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

Example

Then $$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so $$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$

Task

Solve the DEs

1. $\frac{1}{x}\frac{dy}{dx} + y = 1.$

2. $\frac{dy}{dx} - 2(x+1)^3 = \frac{4}{x+1}y$ at $y(0) = 0.$

Answer

Re-arranging

$$\frac{dy}{dx} - 2(x+1)^3 = \frac{4}{x+1}y$$

$$\frac{dy}{dx} - \frac{4}{x+1}y = 2(x+1)^3$$

$\lambda = e^{\int P(x)dx} = e^{\int \frac{-4}{x+1}dx} = e^{-4\int \frac{1}{x+1}dx} = e^{-4(\ln|x+1|)} = e^{\ln(x+1)^{-4}} = \frac{1}{(x+1)^4}.$

Multiplying by λ we get

$$\left(\frac{1}{(x+1)^4}\right)\frac{dy}{dx} - \left(\frac{1}{(x+1)^4}\right)\frac{4}{x+1}y = 2\left(\frac{1}{(x+1)^4}\right)(x+1)^3$$

$$\frac{d}{dx}\left(\frac{y}{(x+1)^4}\right) = \frac{2}{x+1}$$

Integrating both sides (LHS is λy) ALWAYS)

$$\frac{y}{(x+1)^4} = \int \frac{2}{x+1}dx$$

$$\frac{y}{(x+1)^4} = 2\ln|x+1| + c$$

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

2.

- What differential equation does the function $P(t)$ satisfy? $\frac{dP(t)}{dt} = kP(t)$
- What is the value of k? k = rate of growth = 0.1
- What is $P(0)$? P(0) = initial pop. size = 500
- Give a formula for $P(t)$. $P(t) = P(0)e^{kt} = 500e^{0.1t}$
- What will the population be at the end of the year 2050? At the end of the year 2050 we will have $t = 50$ and the population will be

$$P(50) = 500e^{0.1(50)} = 500e^{5} \approx 74,206$$

3.

We have

$$\begin{aligned} 1,000,000 &= Pe^{0.05(40)} \\ P &= 135,335.28. \end{aligned}$$

She must invest $\$135,335.28$ at 5 interest.

4.

2. If some roots are repeated, e.g. λ_1 has multiplicity k_1, λ_2 has multiplicity k_2 etc. you must multiply the corresponding exponential with a polynomial of the order $k_1 - 1, k_2 - 1$ etc. with arbitrary coefficients. The solution then looks like this:

$$y = \left(C_1 + C_2 x + \ldots C_{k_1} x^{k_1 - 1}\right) e^{\lambda_1 x} + \left(D_1 + D_2 x + \ldots D_{k_2} x^{k_2 - 1}\right) e^{\lambda_2 x} + \ldots$$

Note that in either case we will have the right number (i.e. *n*) of the arbitrary constants.

At this *level*, we restrict our attention to second-order linear homogeneous differential equations with ***constant coefficients*** only.

Second-order linear equations

A second-order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions.

In this section we study the case where $G(x) = 0$, for all x, in the ODE. Such equations are called **homogeneous linear** equations. Or we can rewrite:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$ (basic form)

where *a, b,* and *c* are constants.

Replacing $\frac{d^2y}{dx}$ with m^2, $\frac{dy}{dx}$ with m, and y with 1 will result

$am^2 + bm + c = 0$ => is called "*auxiliary quadratic equation*"

or "*auxiliary equation*".

Thus, the general solution of the 2nd – order linear differential equation

Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By (10), the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$ □

Example

Solve $y''' - 9y'' + 20y = 0$.

Solution

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, which can be factored into

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The roots are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$; hence the solution is

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

Case III: $b^2 - 4ac < 0$

In this case the roots and of the auxiliary equation are complex numbers.

If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Example