Basic Mathematics

Example 2.1: Simplify the given expressions:

(i) $-3x - (5 - 4x)$.

(ii) $\frac{1}{5}x - \frac{3}{4}x$.

(iii) $-4(x - 5) + 3(4 - 2x)$.

(iv) $2a^2 - b + 3(a^2 - b)$.

Solutions:

(i) $-3x - (5 - 4x) = -3x - 5 + 4x = x(-3 + 4) - 5 = x - 5$.

(ii) $\frac{1}{5}x - \frac{3}{4}x = x(\frac{1}{5} - \frac{3}{4}) = \frac{1 \cdot 4 - 3 \cdot 5}{20}x = -\frac{11}{20}x$.

(iii) $-4(x - 5) + 3(4 - 2x) = -4x + 20 + 12 - 6x = -10x + 32$.

(iv) $2a^2 - b + 3(a^2 - b) = 2a^2 - b + 3a^2 - 3b = 5a^2 - 4b$.

Remark 2.1

The following is standard procedure with rational exponents:

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

The rules for computing with rational exponents are the same as the ones for integral exponents that we have encountered before.

Polynomials

Definition

A *polynomial* in x is a sum of one or more terms each of which consists of a constant multiplied by a nonnegative integer power of x; the variable x is often called an *indeterminate* in this context. The general form of a polynomial is

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, \ldots, a_n$ are real numbers. They are the *coefficients* of the polynomial.

Definition 2.3

1. A polynomial with only one term is called a *monomial*.
2. A polynomial with two terms is called a *binomial*.
3. A polynomial with three terms is called a *trinomial*.
4. A polynomial of degree 2 is called a *quadratic*.
5. Other terms, such as "cubic", "quartic", "quintic" have been used to denote polynomials of degree 3, 4, and 5.

Example

$1 = 0x + 1$, $3x^2 + 4 = 3x^2 + 0x + 4$, but $3x^2 + 4 \neq 3x^3 + 5$.

A nonzero polynomial is usually written as a sum of terms that are powers of x multiplied by nonzero coeffients (i.e. monomials of the form $0x^k$ are omitted). Two nonzero polynomials written in this way are equal if and only if they have the same degree and corresponding coefficients are equal.

All roots are assumed to be defined.

Rules for radicals

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$ *Product rule*
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ *Quotient rule*

Example 1.10. Simplify

(i) $\sqrt[3]{125\,x^6}$.

(ii) $\sqrt{\dfrac{3}{16}}$.

(iii) $\sqrt[5]{\dfrac{-32y^5}{x^{20}}}$.

(iv) $\sqrt{20}$.

(v) $\dfrac{9}{\sqrt{3}}$.

Solutions:

(i) $(\sqrt{2}-2\sqrt{3})(2+4\sqrt{3})=2\sqrt{2}+4\sqrt{6}-4\sqrt{3}-24=$
$2\sqrt{2}(1+2\sqrt{3}-\sqrt{6}-12\sqrt{2})$.

(ii) $(3-\sqrt{6})(3+\sqrt{6})=3^2-\sqrt{6}^2=9-6=3$.

(iii) $\dfrac{\sqrt{3}}{3-\sqrt{6}}=\dfrac{\sqrt{3}(3+\sqrt{6})}{(3-\sqrt{6})(3+\sqrt{6})}=$
$\dfrac{\sqrt{3}(3+\sqrt{6})}{3^2-6}=\dfrac{3\sqrt{3}+3\sqrt{2}}{3}=\sqrt{3}+\sqrt{2}$.

Solutions:

(i) Here, we have $a_0a_2 = 6$. Divisors of 6 are $-6, -3, -2, -1, 1, 2, 3, 6$. We try $-5 = -3 + (-2)$ and rewrite $x^2 - 5x + 6$ as $x^2 - 2x - 3x + 6$. Factorising the two terms separately, we get $x(x-2) - 3(x-2) = (x-2)(x-3)$.

(ii): Again, $a_0a_2 = 6$. The only way to write 7 as a sum of two divisors of 6 is $7 = 1 + 6$. Now we write $2x^2+7x+3 = 2x^2+6x+x+3 = 2x(x+3)+(x+3) = (2x+1)(x+3)$.

(v) $x^3 - 27 = x^3 - 3^3 = (x-3)(x^2+3x+9)$.

(vi) $8a^6 + 125b^3 = ((2a^2)^3 + (5b)^3) =$
$= (2a+5b)(4a^2 - 20a^2b + 25b^2)$.

(vii) $y^{3m} - 1 = ((y^m)^3 - 1^3) = (y^m - 1)(y^{2m} + y^m + 1)$.

Miscellaneous problems

(i) Let $f(x) = 12x^3 + 25x^2 + 9x - 4$. Factorise $f(x)$ completely and determine the roots.

(ii) Completely factorise the polynomial $4x^4 + 7x^2 - 2$.

(iii) Simplify the expression $\dfrac{3x^2 + (1 + 3\sqrt{2})x + \sqrt{2}}{(x - \sqrt{2})(3x + 1)}$.

(iv) Factorise $x^4 + 8x^3 + 18x^2 + 15x + 5$, given that -1 is a root.

(ii) The polynomial $4x^4 + 7x^2 - 2$ has degree four; but, as it only involves even powers of x, we can work with a *substitution*. Letting $x^2 = u$, we try to factorise the polynomial $4u^2 + 7u - 2$.
This yields $4u^2 + 7u - 2 = 4u^2 + 8u - u - 2 = 4u(u + 2) - u - 2 = (u + 2)(4u - 1)$.

Putting x back where it belongs, we obtain $4x^4 + 7x^2 - 2 = (x^2 + 2)(4x^2 - 1)$. Now -2 is not the square of any real number, so $x^2 + 2$ is irreducible; however, $4x^2 - 1 = ((2x)^2 - 1^2) = (2x + 1)(2x - 1)$. Hence the complete factorisation is $4x^4 + 7x^2 - 2 = (x^2 + 2)(2x + 1)(2x - 1)$.