Instead of a_k we usually use p_k, the probability that a random variable X takes a non-negative integer. The p.g.f of a random variable is usually denoted by $P(s)$ or $G(s)$.

Definition 2

The generating function $G(s)$ of the integer-valued random X is defined by

$$G(s) = E[S^X] = \sum_{k=0}^{\infty} Prob(X = k)s^k$$

Deriving moments of random variable using p.g.f

From $G(s) = E[S^X]$,the first and second derivatives of $G(s)$ with respect to s are

$$\begin{aligned}\frac{dG(s)}{ds} &= G'(s) = E[XS^{X-1}] \\ \frac{d^2G(s)}{ds^2} &= G''(s) = E[X(X-1)S^{X-2}]\end{aligned}$$

Substituting $s = 1$, we have

$$\begin{aligned}G(1) &= \sum p_k = 1 \\ G'(1) &= E[X] \\ G''(1) &= E[X(X-1)]\end{aligned}$$

the mean and variance of X are

$$E[X] = G'(1)$$

and

$$\begin{aligned}Var(X) &= E(X^2) - [E(X)]^2 \\ &= E[X(X-1)] + E[X] - [E(X)]^2 \\ &= G''(1) + G'(1) - [G'(1)]^2\end{aligned}$$

Examples of p.g.f:

(i) Let X have binomial distribution with parameters n and p;

$$P(X = x) = \binom{n}{x} p^x(1-p)^{n-x}, \quad x = 0, 1, .., n$$

The p.g.f of X is given by

$$\begin{aligned}G(s) &= \sum_{x=0}^{n} P(X = x)s^x \\ &= \sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x}s^x \\ &= [(1-p) + ps]^n\end{aligned}$$

MOMENT GENERATING FUNCTIONS:

Definition: The moment generating function of a random variable X is defined by

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_x e^{tx} f(x) dx \end{aligned}$$

if X is a continuous random variable and

$$\begin{aligned} \phi(t) &= E[e^{tX}] \\ &= \sum_x e^{tx} P(X = x) dx \end{aligned}$$

if X is a discrete random variable.

Properties of moment generating function

(i) The moment-generating function of X is unique in the sense that, if two random variables X and Y have the same moment generating function(mgf) $M_X(t) = M_Y(t)$, for t in an interval containing 0, then X and Y have the same distribution.

(ii) If X and Y are independent, and $Z = X + Y$; then $M_Z(t) = M_X(t) \times M_Y(t)$. That is, the mgf of the sum of two independent random variables is the product of the mgfs of the individual random variables. The result can be extended to several random variables.

(iii) Let $Y = aX + b$. Then $M_Y(t) = e^{bt} M_X(at)$.

Deriving moments using moment generating function

An advantage of the moment generating function is its ability to give the moments of the random variable. We will make use of tthe Maclaurin series. The Maclaurin series of the function e^{tx} is given by

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots \frac{(tx)^n}{n!} + \ldots;$$

By using the fact that the expected value of the sum equals the sum of the expected values, the moment-generating function can be written as

$$\begin{aligned} E[e^{tX}] &= E[1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \ldots \frac{(tX)^n}{n!} + \ldots] \\ &= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] \ldots + \frac{t^n}{n!}E[X^n] + \ldots \end{aligned}$$