Equations with non-constant coefficients with missing y-term.

If the y-term is missing in a 2nd-order linear DE, then the equation can be converted into a first order linear DE and solved using the integrating factor method.

Example:

$$xy'' + 4y' = x^2$$ (Note: this ODE is NOT homogeneous)

Std form: $y'' + \frac{4}{x}y' = x$.

let $u = y'$

then $u' + \frac{4}{x}u = x$ ← first order linear DE

here $P(x) = 4/x$, $Q(x) = x$.

Integrating factor (IF) $= e^{\int 4/x\,dx}$.

$$= e^{4\ln x}$$
$$= e^{\ln x^4} = x^4$$

$$\therefore \frac{d}{dx}(x^4u) = x^5$$

$$= \int d(x^4u) = \int x^5\,dx.$$

$$x^4u = \frac{x^6}{6} + C.$$

$$u(x) = \frac{1}{6}x^2 + Cx^{-4}$$

But $y(x) = \int u(x)\,dx = \frac{x^3}{18} + \frac{Cx^{-3}}{-3} + C_2.$

$$= \frac{x^3}{18} - \frac{Cx^{-3}}{3} + C_2.$$