Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Chain rule:

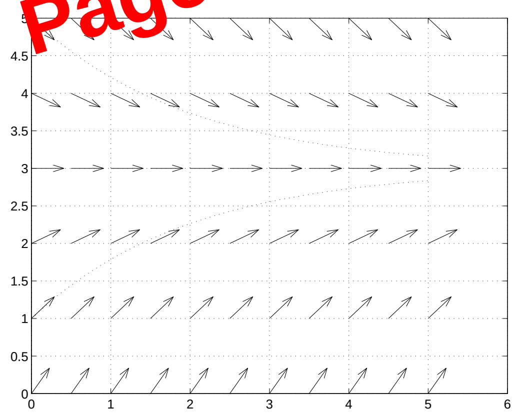
$$\frac{d}{dt}(f(g(t)) = f'(g(t)) \cdot g'(t)$$

Directional field: for first order equations $y' = f(t,y)$.
Interpret y' as the slope of the tangent to the solution $y(t)$ at point (t,y) in the $y - t$ plane.

Example 5. Consider the equation $y' = \dfrac{3-y}{2}$. We know the following:

- If $y = 3$, then $y' = 0$, flat slope,
- If $y > 3$, then $y' < 0$, down slope,
- If $y < 3$, then $y' > 0$, up slope.

See the directional field below (with some solutions sketched):

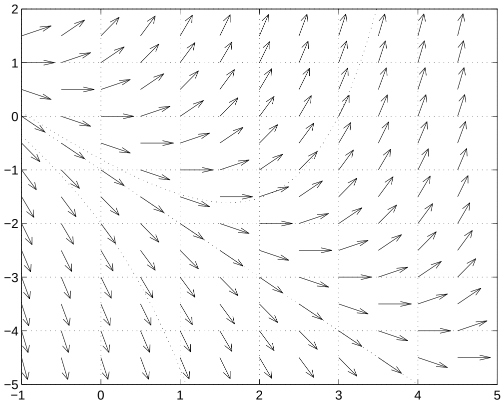


As $t \to \infty$, we have $y \to 3$.

Example 6. $y' = t + y$

- We have $y' = 0$ when $y = -t$,

- We have $y' > 0$ when $y > -t$,
- We have $y' < 0$ when $y < -t$.



What can we say about the solutions?

This depends on the initial condition $y(0) = y_0$.

- If $y(0) > -1$, then $y \to \infty$ as $t \to \pm\infty$.
- If $y(0) < -1$, then $y \to \mp\infty$ as $t \to \pm\infty$.
- If $y(0) = -1$, the $y(t) = -t - 1$.

We note that the statement of this theorem is not as strong as the one for linear equation.

Below we give two counter examples.

Example 1. Loss of uniqueness. Consider

$$\frac{dy}{dy} = f(t,y) = -\frac{t}{y}, \qquad y(-2) = 0.$$

We first note that at $y = 0$, which is the initial value of y, we have $y' = f(t,y) \to \infty$. So the conditions of the Theorem are not satisfied, and we expect something to go wrong.

Solve the equation as an separable equation, we get

$$\int y\,dy = -\int t\,dt, \quad y^2 + t^2 = c$$

and by IC we get $c = (-2)^2 + 0 = 4$, so $y^2 + t^2 = 4$, and $y = \pm\sqrt{4-t^2}$. Both are solutions. We lose uniqueness of solutions.

Example 2. Blow-up of solution. Consider a simple non-linear equation:

$$y' = y^2, \qquad y(0) = 1.$$

Note that $f(t,y) = y^2$, which is defined for all t and y. But, due to the non-linearity of f, solution can not be defined for all t.

This equation can be easily solved as a separable equation.

$$\int \frac{1}{y^2}dy = \int dt, \qquad -\frac{1}{y} = t + c, \quad y(t) = \frac{-1}{t+c}.$$

By IC $y(0) = 1$, we get $1 = -1/(0+c)$, and so $c = -1$, and

$$y(t) = \frac{-1}{t-1}.$$

We see that the solution *blows up* as $t \to 1$, and can not be defined beyond that point.

This kind of blow-up phenomenon is well-known for nonlinear equations.

we get

$$S(t) = 27000e^{0.08t} - 25000.$$

When $t = 40$, we have

$$S(40) = 27000 \cdot e^{3.2} - 25000 \approx 637,378.$$

Compare this to the total amount invested: $2000 + 2000 * 40 = 82,000$.

Example 4: A home-buyer can pay \$800 per month on mortgage payment. Interest rate is 9% annually, (but compounded continuously), mortgage term is 20 years. Determine maximum amount this buyer can afford to borrow.

Answer. Set up the model: Let $Q(t)$ be the amount borrowed (principle) after t years

$$\frac{dQ}{dt} = 0.09Q(t) - 800 * 12$$

The terminal condition is given $Q(20) = 0$. We must find $Q(0)$.

Solve the differential equation:

$$Q' - 0.09Q = -9600, \qquad \mu = e^{-0.09t}$$

$$Q(t) = e^{0.09t} \int (-9600)e^{-0.09t}\,dt = e^{0.09t}\left[-9600\frac{e^{-0.09t}}{-0.09} + c\right] = \frac{9600}{0.09} + ce^{0.09t}$$

By terminal condition,

$$Q(20) = \frac{9600}{0.09} + ce^{0.09*20} = 0, \qquad c = -\frac{9600}{0.09 \cdot e^{1.8}}$$

so we get

$$Q(t) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}}e^{0.09t}.$$

Now we can get the initial amount

$$Q(0) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}} = \frac{9600}{0.09}(1 - e^{-1.8}) \approx 89,034.79.$$

<u>Model III</u>: Mixing Problem.

Example 5. At $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of r gal/min. At the same time, the well-mixed mixture is draining from the tank at the same rate.

Here $g = 9.8$ is the gravity, and $m = 0.5$ is the mass. We have an equation for v:

$$\frac{dv}{dt} = -\frac{1}{10}v - 9.8 = -0.1(v + 98),$$

so

$$\int \frac{1}{v+98}dv = \int (-0.1)dt, \quad \Rightarrow \quad \ln|v + 98| = -0.1t + c$$

which gives

$$v + 98 = \bar{c}e^{-0.1t}, \quad \Rightarrow \quad v = -98 + \bar{c}e^{-0.1t}.$$

By IC:

$$v(0) = -98 + \bar{c} = 10, \quad \bar{c} = 108, \quad \Rightarrow \quad v = -98 + 108e^{-0.1t}.$$

To find the position S, we use $S' = v$ and integrate

$$S(t) = \int v(t)\,dt = \int (-98 + 108e^{-0.1t})dt = -98t + 108e^{-0.1t}/(-0.1) + c$$

By IC for S,

$$S(0) = -1080 + c = 30, \quad c = 1110, \quad S(t) = -98t - 1080e^{-0.1t} + 1110.$$

At the maximum height we have $v = 0$. Let's find out the time T when max height is reached.

$$v(T) = 0, \quad -98 + 108e^{-0.1T} = 0, \quad 98 = 108e^{-0.1T}, \quad e^{-0.1T} = 98/108,$$

$$-0.1T = \ln(98/108), \quad T = -10\ln(98/108) = \ln(108/98).$$

So the max height S_M is

$$\begin{aligned} S_M &= S(T) = -980\ln\frac{108}{98} - 1080e^{-0.1\ln(108/98)} + 1110 \\ &= -980\ln\frac{108}{98} - 1080(98/108) + 1110 \approx 34.78 \text{ m.} \end{aligned}$$

Other possible questions:

- Find the time when the ball hit the ground.
  Solution: Find the time $t = t_H$ for $S(t_H) = 0$.

- Find the speed when the ball hit the ground.
  Solution: Compute $|v(t_H)|$.

- Find the total distance traveled by the ball when it hits the ground.
  Solution: Add up twice the max height S_M with the height of the building.

3.3: Complex Roots

The roots of the characteristic equation can be complex numbers. Consider the equation

$$ay'' + by' + cy = 0, \qquad \rightarrow \quad ar^2 + br + c = 0.$$

The two roots are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, the root are complex, i.e., a pair of complex conjugate numbers. We will write $r_{1,2} = \lambda \pm i\mu$. There are two solutions:

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}, \qquad y_2 = y_1 = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t}.$$

To deal with exponential function with pure imaginary exponent, we need the Euler's Formula:

$$e^{i\beta} = \cos\beta + i\sin\beta.$$

A couple of Examples to practice this formula:

$$e^{i\frac{5}{6}\pi} = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + i\frac{1}{2}.$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b).$$

Back to y_1, y_2, we have

$$y_1 = e^{\lambda t}(\cos\mu t + i\sin\mu t), \qquad y_2 = e^{\lambda t}(\cos\mu t + i\sin\mu t).$$

But these solutions are complex valued. We want real-valued solutions! To achieve this, we use the Principle of Superposition. If y_1, y_2 are two solutions, then $\frac{1}{2}(y_1 + y_2)$, $\frac{1}{2i}(y_1 - y_2)$ are also solutions. Let

$$\tilde{y}_1 \doteq \frac{1}{2}(y_1 + y_2) = e^{\lambda t}\cos\mu t, \qquad \tilde{y}_2 \doteq \frac{1}{2i}(y_1 - y_2) = e^{\lambda t}\sin\mu t.$$

To make sure they are linearly independent, we can check the Wronskian,

$$W(\tilde{y}_1, \tilde{y}_2) = \mu e^{2\lambda t} \neq 0. \quad \text{(home work problem).}$$

Answer. The characteristic equation is

$$r^2 + 2r + 101 = 0, \quad \Rightarrow \quad r_{1,2} = -1 \pm 10i, \quad \Rightarrow \quad \lambda = -1,\ \mu = 10.$$

So the general solution is

$$y(t) = e^{-t}(c_1 \cos 10t + c_2 \sin 10t),$$

so

$$y'(t) = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-10c_1 \sin t + 10c_2 \cos t)$$

Fit in the ICs:

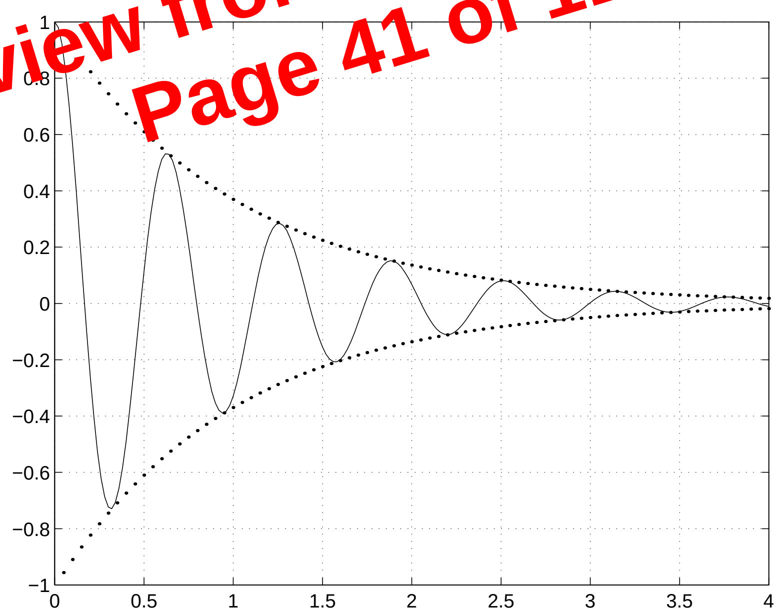
$$y(0) = 1: \qquad y(0) = e^0(c_1 + 0) = c_1 = 1,$$

$$y'(0) = 0: \qquad y'(0) = -1 + 10c_2 = 0, \quad c_2 = 0.1.$$

Solution is

$$y(t) = e^{-t}(\cos t + 0.1 \sin t).$$

The graph is given below:



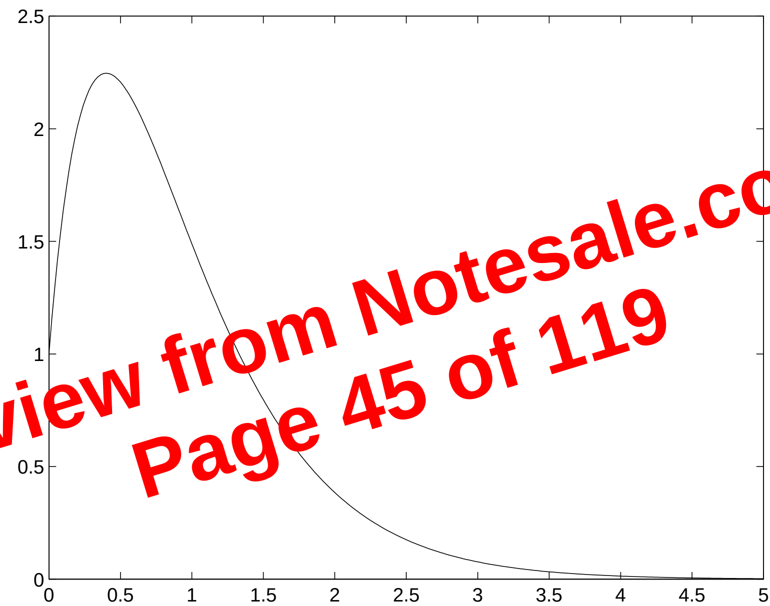
We see it is a decaying oscillation. The sin and cos part gives the oscillation, and the e^{-t} part gives the decaying amplitude. As $t \to \infty$, we have $y \to 0$.

Cancel the term e^{-2t}, and we get $v'' = 0$, which gives $v(t) = c_1 t + c_2$. So

$$y_2(t) = vy_1 = (c_1 t + c_2)e^{-2t} = c_1 te^{-2t} + c_2 e^{-2t}.$$

Note that the term $c_2 e^{-2t}$ is already contained in cy_1. Therefore we can choose $c_1 = 1, c_2 = 0$, and get $y_2 = te^{-2t}$, which gives the same general solution as Method 1. We observe that this method involves more computation than Method 1.

A typical solution graph is included below:



We see if $c_2 > 0$, y increases for small t. But as t grows, the exponential (decay) function dominates, and solution will go to 0 as $t \to \infty$.

One can show that in general if one has repeated roots $r_1 = r_2 = r$, then $y_1 = e^{rt}$ and $y_2 = te^{rt}$, and the general solution is

$$y = c_1 e^{rt} + c_2 tr^{rt} = e^{rt}(c_1 + c_2 t).$$

Example 2. Solve the IVP

$$y'' - 2y' + y = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

Solve this for y_2:

$$\mu = \exp(\int \frac{1}{t} dt) = \exp(\ln t) = t, \quad \Rightarrow \quad y_2 = \frac{1}{t} \int t \cdot t^{-\frac{3}{2}} dt = \frac{1}{t}(\frac{2}{3} t^{\frac{3}{2}} + C).$$

Let $C = 0$, we get $y_2 = \frac{2}{3}\sqrt{t}$. Since $\frac{2}{3}$ is a constant multiplication, we can drop it and choose $y_2 = \sqrt{t}$.

Method 2: This is the textbook's version. We saw in the previous example that this method is inferior to Method 1, therefore we will not focus on it at all. If you are interested in it, read the book.

Let's introduce another method that combines the ideas from Method 1 and Method 2.

Method 3. We will use Abel's Theorem, and at the same time we will seek a solution of the form $y_1 = vy_1$.

By Abel's Theorem, we have (worked out in M1) $W(y_1, y_2) = t^{-\frac{3}{2}}$. Now, seek $y_2 = vy_1$. By the definition of the Wronskian, we have

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1 (vy_1)' - y_1'(vy_1) = y_1(v' y_1 + v y_1'') - v y_1 y_1' = v' y_1^2.$$

Note that this is a general formula.

Now putting $y_1 = 1/t$, we get

$$v' \frac{1}{t^2} = t^{-\frac{3}{2}}, \qquad v' = t^{\frac{1}{2}}, \qquad v = \int t^{\frac{1}{2}} dt = \frac{2}{3} t^{\frac{3}{2}}.$$

Drop the constant $\frac{2}{3}$, we get

$$y_2 = v y_1 = t^{\frac{3}{2}} \frac{1}{t} = t^{\frac{1}{2}}.$$

We see that Method 3 is the most efficient one among all three methods. We will focus on this method from now on.

Example 4. Consider the equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0.$$

So: we have $mg = kL$ which give

$$k = \frac{mg}{L}$$

which gives a way to obtain k by experiment: hang a mass m and measure the elongation L.

Model the motion: Let $u(t)$ be the displacement/position of the mass at time t, assuming the origin $u = 0$ at the equilibrium position, and downward the positive direction.
Total elongation: $L + u$
Total spring force: $F_s = -k(L + u)$

Other forces:
* damping/resistent force: $F_d(t) = -\gamma v = -\gamma u'(t)$, where γ is the damping constant, and v is the velocity
* External force applied on the mass: $F(t)$, given function of t.

Total force on the mass: $\sum f = mg + F_s + F_d + F$.
Newton's law of motion $ma = \sum f$ gives

$$ma = mu'' = \sum f = mg + F_s + F_d + F \quad \Rightarrow \quad mu'' = mg - k(L+u) - \gamma u' + F.$$

Since $mg = kL$, by rearranging the terms, we get

$$mu'' + \gamma u' + ku = F$$

where m ia the mass, γ is the damping constant, k is the spring constant, and F is the external force.

Next we study several cases.

Case 1: Undamped free vibration (simple harmonic motion). We assume no damping $(\gamma = 0)$ and no external force $(F = 0)$. So the equation becomes

$$mu'' + ku = 0.$$

Solve it

$$mr^2 + k = 0, \quad r^2 = -\frac{k}{m}, \quad r_{1,2} = \pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i, \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}}.$$

Inverse Laplace transform. Definition:

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \text{if} \qquad F(s) = \mathcal{L}\{f(t)\}.$$

Technique: find the way back.

Some simple examples:

Example 10.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t.$$

Example 11.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\} = \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}e^{-5t}t^3.$$

Example 12.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t\frac{1}{2}\sin 2t.$$

Example 13.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/4}{s-2}+\frac{1/4}{s+2}\right\} = \frac{3}{4}e^{2t}+\frac{1}{4}e^{-2t}.$$

Here we used partial fraction to find out:

$$\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2}+\frac{B}{s+2}, \qquad A = 3/4, \quad B = 1/4.$$

$$(s^2+1)Y(s) = \frac{s}{s^2+4} + 2s + 1 = \frac{2s^3+s^2+9s+4}{s^2+4}$$

$$Y(s) = \frac{2s^3+s^2+9s+4}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

Comparing numerators, we get

$$2s^3+s^2+9s+4 = (As+B)(s^2+4) + (Cs+D)(s^2+1).$$

One may expand the right-hand side and compare terms to find A, B, C, D, but that takes more work.

Let's try by setting s into complex numbers.

Set $s = i$, and remember the facts $i^2 = -1$ and $i^3 = -i$, we have

$$-2i - 1 + 9i + 4 = (Ai + B)(-1+4),$$

which gives

$$3 + 7i = 3B + 3Ai \quad \Rightarrow \quad B = 1, \quad A = \frac{7}{3}.$$

Set now $s = 2i$:

$$-16i - 4 + 18i + 4 = (2Ci + D)(-3),$$

then

$$0 + 2i = -3D - 6Ci, \quad \Rightarrow \quad D = 0, C = -\frac{1}{3}.$$

So

$$Y(s) = \frac{7}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

and

$$y(t) = \frac{7}{3}\cos t + \sin t - \frac{1}{3}\cos 2t.$$

A very brief review on partial fraction, targeted towards inverse Laplace transform.

Goal: rewrite a fractional form $\frac{P_n(s)}{P_m(s)}$ (where P_n is a polynomial of degree n) into sum of "simpler" terms. We assume $n < m$.

Compare s^2-term: $0 = A + B$, so $B = -A = -1$.

Compare s-term: $0 = A + C$, so $C = -A = -1$.

So

$$Y(s) = e^{-s}\left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right) + \frac{s+1}{s^2+s+1}.$$

We work out some detail

$$\frac{s+1}{s^2+s+1} = \frac{s+1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2},$$

so

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$$

We conclude

$$\begin{aligned} y(t) &= u_1(t)\left[1 - e^{-\frac{1}{2}(t-1)}\left(\cos\frac{\sqrt{3}}{2}(t-1) - \sin\frac{\sqrt{3}}{2}(t-1)\right)\right] \\ &\quad + e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]. \end{aligned}$$

Remark: There are other ways to work out the partial fractions.

Extra question: What happens when $t \to \infty$?

Answer: We see all the terms with the exponential function will go to zero, so $y \to 1$ in the limit. We can view this system as the spring-mass system with damping. Since $g(t)$ becomes constant 1 for large t, and the particular solution (which is also the steady state) with 1 on the right hand side is 1, which provides the limit for y.

Further observation:

- We see that the solution to the homogeneous equation is

$$e^{-\frac{1}{2}t}\left[c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right],$$

and these terms do appear in the solution.

- Actually the solution consists of two part: the forced response and the homogeneous solution.
- Furthermore, the g has a discontinuity at $t = 1$, and we see a jump in the solution also for $t = 1$, as in the term $u_1(t)$.

Example 2. (Undamped system with force, pure imaginary roots) Solve the following initial value problem

$$y'' + 4y = g(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 1, & \pi \le t < 2\pi, \\ 0, & 2\pi \le t, \end{cases} \qquad y(0) = 1, \quad y'(0) = 0\,.$$

Rewrite

$$g(t) = u_\pi(t) - u_{2\pi}(t), \qquad \mathcal{L}\{g\} = e^{-\pi s}\frac{1}{s} - e^{-2\pi}\frac{1}{s}.$$

So

$$s^2 Y - s + 4Y = \frac{1}{s}\left(e^{-\pi s} - e^{-2\pi s}\right).$$

Solve it for Y:

$$Y(s) = \frac{e^{-\pi} - e^{-2\pi}}{s(s^2+4)} + \frac{s}{s^2+4} = \frac{e^{-\pi}}{s(s^2+4)} - \frac{e^{-2\pi}}{s(s^2+4)} + \frac{s}{s^2+4}\,.$$

Work out partial fraction

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2+4}, \qquad A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = 0.$$

So

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+4)}\} = \frac{1}{4} - \frac{1}{4}\cos 2t\,.$$

Now we take inverse Laplace transform of Y

$$\begin{aligned} y(t) &= u_\pi(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right) - u_{2\pi}(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-2\pi)\right) + \cos 2t \\ &= (u_\pi(t) - u_{2\pi})\frac{1}{4}(1 - \cos 2t) + \cos 2t \\ &= \cos 2t + \begin{cases} \frac{1}{4}(1-\cos 2t), & \pi \le t < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \\ &= \text{homogeneous solution} + \text{forced response} \end{aligned}$$

Example 3. In Example 2, let

$$g(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^t, & 4 \le 5 < 2\pi, \\ 0, & 5 \le t. \end{cases}$$

Find $Y(s)$.

Answer. Rewrite

$$g(t) = e^t(u_4(t) - u_5(t)) = u_4(t)e^{t-4}e^4 - u_5(t)e^{t-5}e^5 \,,$$

so

$$G(s) = \mathcal{L}\{g(t)\} = e^4 e^{-4s}\frac{1}{s-1} - e^5 e^{-5s}\frac{1}{s-1}\,.$$

Take Laplace transform of the equation, we get

$$(s^2+4)Y(s) = G(s)+s, \qquad Y(s) = \left(e^4e^{-4s} - e^5e^{-5s}\right)\frac{1}{(s-1)(s^2+4)} + \frac{s}{s^2+4}\,.$$

Remark: We see that the first term will give the forced response, and the second term is from the homogeneous equation.

The students may work out the inverse transform as a practice.

Example 4. (Undamped system with force, example 2 from the book p. 334)

$$y'' + 4y = g(t), \quad y(0) = 0, y'(0) = 0, \quad g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le 5 < 10, \\ 1, & 10 \le t. \end{cases}$$

Let's first work on $g(t)$ and its Laplace transform

$$g(t) = \frac{t-5}{5}(u_5(t) - u_{10}(t)) + u_{10}(t) = \frac{1}{5}u_5(t)(t-5) - \frac{1}{5}u_{10}(t)(t-10),$$

$$G(s) = \mathcal{L}\{g\} = \frac{1}{5}e^{-5s}\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$$

Let $Y(s) = \mathcal{L}\{y\}$, then

$$(s^2+4)Y(s) = G(s), \qquad Y(s) = \frac{G(s)}{s^2+4} = \frac{1}{5}e^{-5s}\frac{1}{s^2(s^2+4)} - \frac{1}{5}e^{-10s}\frac{1}{s^2(s^2+4)}$$

we get

$$\begin{cases} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ \vdots & \\ x_{n-1}' &= y^{(n-1)} = x_n \\ x_n' &= y^{(n)} = F(t, x_1, x_2, \cdots, x_n) \end{cases}$$

with corresponding source terms.

Reversely, we can convert a 1st order system into a high order equation.

Example 2. Given

$$\begin{cases} x_1' &= 3x_1 - 2x_2 \\ x_2' &= 2x_1 - 2x_2 \end{cases} \qquad \begin{cases} x_1(0) &= 3 \\ x_2(0) &= \frac{1}{2} \end{cases}$$

Eliminate x_2: the first equation gives

$$2x_2 = 3x_1 - x_1', \qquad x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'$$

Plug this into second equation, we get

$$\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right)' = 2x_1 - 2x_2 = -x_1 + x_1'$$

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = -x_1 + x_1'$$

$$x_1'' - x_1' - 2x_1 = 0$$

with the initial conditions:

$$x_1(0) = 3, \quad x_1'(0) = 3x_1(0) - 2x_2(0) = 8.$$

This we know how to solve!

Definition of a solution: a set of functions $x_1(t), x_2(t), \cdots, x_n(t)$ that satisfy the differential equations and the initial conditions.

Definition: If $\lambda_1 \neq \lambda_2$ are real with the same sign, the critical point $\vec{x} = 0$ is called a *node*.

If $\lambda_1 > 0$, $\lambda_2 > 0$, this node is called a *source*.

If $\lambda_1 < 0$, $\lambda_2 < 0$, this node is called a *sink*.

A sink is stable, and a source is unstable.

Example 4. (Source node) Suppose we know the eigenvalues and eigenvectors of A are

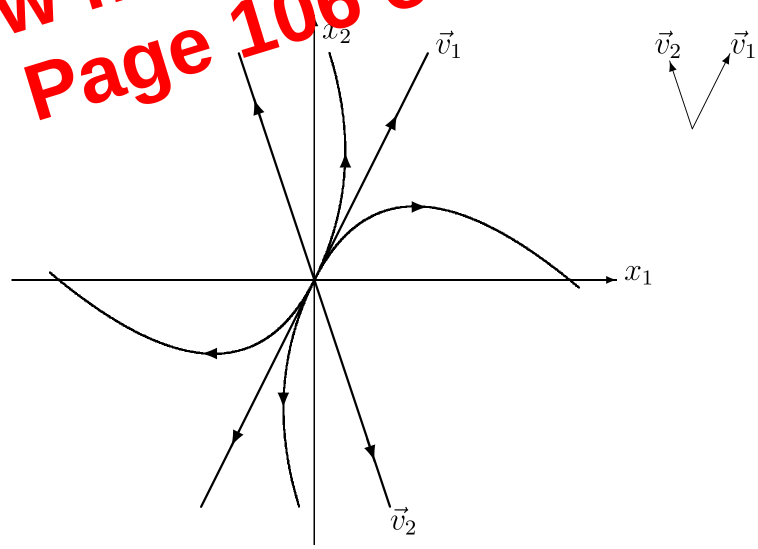
$$\lambda_1 = 3, \quad \lambda_2 = 4, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(1) Find the general solution for $\vec{x}' = A\vec{x}$, (2) Sketch the phase portrait.

Answer. (1) The general solution is simple, just use the formula

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(2) Phase portrait: Since $\lambda_2 > \lambda_1$, then the solution approach $\vec{v}_2$ as time grows. As $t \to -\infty$, $\vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1$. See the plot below.



Summary:
(1). If λ_1 and λ_2 are real and with opposite sign: the origin is a saddle point,

7.8: Repeated eigenvalues

Here we study the case where the two eigenvalues are the same, say $\lambda_1 = \lambda_2 = \lambda$. This can happen, as we will see through our first example.

Example 1. Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then

$$\det(A-\lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2-4\lambda+3+1 = (\lambda-2)^2 = 0,$$

so $\lambda_1 = \lambda_2 = 2$. And we can find only one eigenvector $\vec{v} = (a,b)^T$

$$(A-\lambda I)\vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \rightarrow \quad a + b = 0.$$

Choosing $a = 1$, then $b = -1$ and we find $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, one solution is:

$$\vec{z}_1 = e^{\lambda t}\vec{v} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We need to find a second solution. Let's try $\vec{z}_2 = te^{\lambda t}\vec{v}$. We have

$$\vec{z}' = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} = (1+\lambda t)e^{\lambda t}\vec{v}$$

$$A\vec{z}_2 = Ate^{\lambda t}\vec{v} = te^{\lambda t}(A\vec{v}) = te^{\lambda t}\lambda\vec{v} = \lambda t e^{\lambda t}\vec{v}$$

If $\vec{z}_2$ is a solution, we must have

$$\vec{z}' = A\vec{z} \quad \rightarrow \quad 1 + \lambda t = \lambda t$$

which doesn't work.

Try something else: $\vec{z}_2 = te^{\lambda t}\vec{v} + \vec{\eta}e^{\lambda t}$. (here $\vec{\eta}$ is a constant vector to be determined later). Then

$$\vec{z}_2' = (1+\lambda t)e^{\lambda t}\vec{v} + \lambda\vec{\eta}e^{\lambda t} = \lambda t e^{\lambda t}\vec{v} + e^{\lambda t}(\vec{v} + \lambda\vec{\eta})$$

$$A\vec{z}_2 = \lambda t e^{\lambda t}\vec{v} + A\vec{\eta}e^{\lambda t}.$$