is similar and left to the reader). For $0 < c < 1$, define the (measurable) sets $\{A_n\}_{n\in\mathbb{N}}$ by

$$A_n = \{f_n \geq cg\}, n \in \mathbb{N}.$$

By the increase of the sequence $\{f_n\}_{n\in\mathbb{N}}$, the sets $\{A_n\}_{n\in\mathbb{N}}$ also increase. Moreover, since the function $cg$ satisfies $cg(x) \leq g(x) \leq f(x)$ for all $x \in S$ and $cg(x) < f(x)$ when $f(x) > 0$, the increasing convergence $f_n \to f$ implies that $\cup_n A_n = S$. By non-negativity of $f_n$ and monotonicity,

$$\int f_n \, d\mu \geq \int f_n \mathbf{1}_{A_n} \, d\mu \geq c \int g \mathbf{1}_{A_n} \, d\mu,$$

and so

$$\sup_n \int f_n \, d\mu \geq c \sup_n \int g \mathbf{1}_{A_n} \, d\mu.$$

Let $g = \sum_{i=1}^k \alpha_i \mathbf{1}_{B_i}$ be a simple-function representation of $g$. Then

$$\int g \mathbf{1}_{A_n} \, d\mu = \int \sum_{i=1}^k \alpha_i \mathbf{1}_{B_i \cap A_n} \, d\mu = \sum_{i=1}^k \alpha_i \mu(B_i \cap A_n).$$

Since $A_n \nearrow S$, we have $A_n \cap B_i \nearrow B_i$, $i = 1, \dots, k$, and the continuity of measure implies that $\mu(A_n \cap B_i) \nearrow \mu(B_i)$. Therefore,

$$\int g \mathbf{1}_{A_n} \, d\mu \nearrow \sum_{i=1}^k \alpha_i \mu(B_i) = \int g \, d\mu.$$



Consequently,

$$\lim_n \int f_n \, d\mu = \sup_n \int f_n \, d\mu \geq c \int g \, d\mu, \text{ for all } c \in (0,1),$$

and the proof is completed when we let $c \to 1$. □

*Remark* 3.9.

1. The monotone convergence theorem is really about the robustness of the Lebesgue integral. Its stability with respect to limiting operations is one of the reasons why it is a de-facto "industry standard".

2. The "monotonicity" condition in the monotone convergence theorem cannot be dropped. Take, for example $S = [0,1]$, $\mathcal{S} = \mathcal{B}([0,1])$, and $\mu = \lambda$ (the Lebesgue measure), and define

$$f_n = n \mathbf{1}_{(0,n^{-1}]}, \text{ for } n \in \mathbb{N}.$$

Then $f_n(0) = 0$ for all $n \in \mathbb{N}$ and $f_n(x) = 0$ for $n > \frac{1}{x}$ and $x > 0$. In either case $f_n(x) \to 0$. On the other hand

$$\int f_n \, d\lambda = n\lambda\left((0, \tfrac{1}{n}]\right) = 1,$$

## *Additional Problems*

**Problem 3.11** (The monotone-class theorem)**.** Prove the following result, known as the *monotone-class theorem* (remember that $a_n \nearrow a$ means that $a_n$ is a non-decreasing sequence and $a_n \to a$)

*Hint:* Use Theorems 3.10 and 2.12

> *Let $\mathcal{H}$ be a class of bounded functions from $S$ into $\mathbb{R}$ satisfying the following conditions*
>
> 1. *$\mathcal{H}$ is a vector space,*
> 2. *the constant function $1$ is in $\mathcal{H}$, and*
> 3. *if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative functions in $\mathcal{H}$ such that $f_n(x) \nearrow f(x)$, for all $x \in S$ and $f$ is bounded, then $f \in \mathcal{H}$.*
>
> *Then, if $\mathcal{H}$ contains the indicator $\mathbf{1}_A$ of every set $A$ in some $\pi$-system $\mathcal{P}$, then $\mathcal{H}$ necessarily contains every bounded $\sigma(\mathcal{P})$-measurable function on $\mathcal{S}$.*

**Problem 3.12** (A form of continuity for Lebesgue integration)**.** Let $(S, \mathcal{S}, \mu)$ be a measure space, and suppose that $f \in \mathcal{L}^1$. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{S}$ and $\mu(A) < \delta$, then $\left|\int_A f\, d\mu\right| < \varepsilon$.



**Problem 3.13** (Sums as integrals)**.** In the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, let $\mu$ be the counting measure.

1. For a function $f : \mathbb{N} \to [0, \infty]$, show that

$$\int f\, d\mu = \sum_{n=1}^{\infty} f(n).$$

2. Use the monotone convergence theorem to show the following special case of Fubini's theorem

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{kn},$$

whenever $\{a_{kn} : k, n \in \mathbb{N}\}$ is a double sequence in $[0, \infty]$.

3. Show that $f : \mathbb{N} \to \mathbb{R}$ is in $\mathcal{L}^1$ if and only if the series

$$\sum_{n=1}^{\infty} f(n),$$

converges absolutely.