FOURIER SERIES

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A self-contained Tutorial Module for learning the technique of Fourier series analysis

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● This property of repetition defines a **fundamental spatial frequency** $k = \frac{2\pi}{L}$ that can be used to give a **first approximation** to the periodic pattern $f(x)$:

$$f(x) \simeq c_1 \sin(kx + \alpha_1) = a_1 \cos(kx) + b_1 \sin(kx),$$

where symbols with subscript 1 are constants that determine the amplitude and phase of this first approximation

● A much **better approximation** of the periodic pattern $f(x)$ can be built up by adding an appropriate combination of **harmonics** to this fundamental (sine-wave) pattern. For example, adding

$$\begin{array}{ll} c_2 \sin(2kx + \alpha_2) = a_2 \cos(2kx) + b_2 \sin(2kx) & \quad \text{(the 2nd harmonic)} \\ c_3 \sin(3kx + \alpha_3) = a_3 \cos(3kx) + b_3 \sin(3kx) & \quad \text{(the 3rd harmonic)} \end{array}$$

Here, symbols with subscripts are constants that determine the amplitude and phase of each harmonic contribution

● In this Tutorial, we consider working out Fourier series for functions $f(x)$ with period $L = 2\pi$. Their fundamental frequency is then $k = \frac{2\pi}{L} = 1$, and their Fourier series representations involve terms like

$$\begin{array}{ll} a_1 \cos x \ , & b_1 \sin x \\ a_2 \cos 2x \ , & b_2 \sin 2x \\ a_3 \cos 3x \ , & b_3 \sin 3x \end{array}$$

We also include a constant term $a_0/2$ in the Fourier series. This allows us to represent functions that are, for example, entirely above the $x-$axis. With a sufficient number of harmonics included, our approximate series can exactly represent a given function $f(x)$

$$\begin{aligned} f(x) = a_0/2 \quad & + \quad a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ... \\ & + \quad b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ... \end{aligned}$$

Exercise 2.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \ . \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right]$$
$$+ \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - ...\right]$$

c) By giving appropriate values to x, show that

(i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ and (ii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

Exercise 4.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \frac{x}{2} \text{ over the interval } 0 < x < 2\pi.$$

a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{\pi}{2} - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x$$

$$\text{i.e. } a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \;=\; \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,\mathrm{d}x\right\}$$

(using integration by parts)

$$\begin{aligned}
\text{i.e. } a_n &= \frac{1}{\pi}\left\{\left(\pi\frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{(\quad 0 \quad - \; 0) \; + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi n^2}\left\{\cos n\pi - \cos 0\right\} = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\} \\
\text{i.e. } a_n &= \begin{cases} 0 & ,\, n \text{ even} \\ -\frac{2}{\pi n^2} & ,\, n \text{ odd} \end{cases} \text{ , see } \textsc{Trig}.
\end{aligned}$$

b) Fourier series representation of $f(x)$

STEP ONE

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} + 0 \\
&= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right] \\
\text{i.e. } a_0 &= \frac{\pi}{2}.
\end{aligned}
$$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot \mathrm{d}x \\
\text{i.e. } a_n &= \frac{1}{\pi}\underbrace{\left\{\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}(-1)\cdot\frac{\sin nx}{n}\,\mathrm{d}x\right\}}_{\text{using integration by parts}} + 0 \\
&= \frac{1}{\pi}\left\{(0-0) + \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right\} \quad \text{, see TRIG} \\
&= \frac{1}{\pi n}\left[\frac{-\cos nx}{n}\right]_0^{\pi} \\
&= -\frac{1}{\pi n^2}\left(\cos n\pi - \cos 0\right) \\
\text{i.e. } a_n &= -\frac{1}{\pi n^2}\left((-1)^n - 1\right) \quad \text{, see TRIG}
\end{aligned}
$$

b) Fourier series representation of $f(x)$

STEP ONE

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2}-\frac{\pi^2}{2}\right) \\ \text{i.e. } a_0 &= 0. \end{aligned}$$

We thus have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

with $a_0 = 0$, $a_n = 0$, $b_n = -\frac{2}{n}(-1)^n$

and

n	1	2	3
b_n	2	-1	$\frac{2}{3}$

Therefore

$$\begin{aligned} f(x) &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots \\ \text{i.e. } f(x) &= 2\left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \ldots \right] \end{aligned}$$

and we have found the required Fourier series.

$$\begin{aligned}\text{i.e. } a_n &= \frac{-2}{n\pi}\left\{-\frac{2\pi}{n}(-1)^n\right\}\\ &= \frac{+4\pi}{\pi n^2}(-1)^n\\ &= \frac{4}{n^2}(-1)^n\end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even}\\ \frac{-4}{n^2} & , n \text{ odd.}\end{cases}$$

The graph of $f(x)$ gives that $f(\pi) = \pi^2$ and the series converges to this value.

Setting $x = \pi$ in the Fourier series thus gives

$$
\begin{aligned}
\pi^2 &= \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi - \frac{1}{4^2}\cos 4\pi + \dots\right) \\
\pi^2 &= \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots\right) \\
\pi^2 &= \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\
\frac{2\pi^2}{3} &= 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\
\text{i.e. } \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots
\end{aligned}
$$

Return to Exercise 7