❖ Property 3

If $\mathcal{L}^{-1}G(s) = g(t)$, then $\mathcal{L}^{-1}\{\frac{G(s)}{s}\} = \int_0^t g(t)\,dt$

❖ Property 4

If $\mathcal{L}^{-1}G(s) = g(t)$, then $\mathcal{L}^{-1}\{e^{-at}\,G(s)\} = u(t-a) \times g(t-a)$

RULES: To invert a transform that contains e^{-as}: $Y(s) = \{e^{-at}\,G(s)\}$

1. Invert G(s) in the usual manner to find g(t).
2. Find $y(t) = g(t-a) \times u(t-a)$ by replacing the *t* arguments, whenever it appears in g(t), by (t - a); then multiply the entire function by the shifted unit step function.

❖ Property 5

- If the Laplace transform contains the factor *s*, the inverse of that transform can be found by suppressing the factor *s*, determining the inverse of the remaining portion of the transform and finally differentiating the inverse with respect to *t*.
- If $\mathcal{L}\{t\,f(t)\} = F'(s)$, then $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{t}\,\mathcal{L}^{-1}\{F'(s)\}$
- If $\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(s)\,ds$, then $f(t) = t\,\mathcal{L}^{-1}\{\int_0^\infty F(s)\,ds\}$

METHOD OF CONVOLUTION

One of the important theorems of Laplace transform is the convolution theorem. It is used in constructing inverses especially for linear differential equations with constant coefficients.

The function, $(f * g)(t) = \int_0^t f(t-\xi)\,g(\xi)\,d\xi$

is called the **convolution** of the functions f and g.

PROPERTIES:

$f * g = g * f$	Commutative
$f * (g * h) = (f * g) * h$	Associative
$f * (g + h) = f * g + f * h$	Distributive

- Theorem 1

If F(s) and G(s) are the Laplace transforms of f(t) and g(t) respectively, then the Laplace transform of the convolution f * g is the product F(s) G(s).

- Theorem 2 **[Convolution or "Faltung" integral]**

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f * g)(t)$$
$$= \int_0^t f(u)\,g(t-u)\,du$$

INVERSE LAPLACE TRANSFORM OF AN INTEGRAL

When g(t) = 1 and $\mathcal{L}\{g(t)\} = G(s) = 1 / s$, the convolution theorem implies that the Laplace transform of the integral of *f* is:

$$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}$$

The inverse form is:

$$\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)\,d\tau$$

Integral Equation

- an equation in which the unknown, call it y(t), occurs in the integrand pf an integral (and may also occur outside the integral)
- can be solved by Laplace transformations if the integral can be written as a convolution

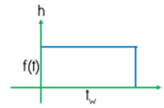
Volterra Integral Equation

$$f(t) = g(t) + \int_0^t f(\tau)\,h(t-\tau)\,d\tau$$

The functions g(t) and h(t) are known.

IMPULSES AND DISTRIBUTION

THE RECTANGULAR PULSE FUNCTION



where: h = height
t_w = width

The flow rate would be held at *h* for duration of t_w units of time. The area under the curve could be interpreted as the material delivered to the tank (= ht_w). Mathematically, the function f(t) is defined as

$$f(t) = \begin{cases} 0 & t < 0 \\ h & 0 \leqslant t < t_w \\ 0 & t \geqslant t_w \end{cases}$$

The Laplace transform of the rectangular pulse can be derived by evaluating the integral between t = 0 and t = t_w because f(t) is zero everywhere else:

$$F(s) = \int_0^\infty e^{-st}\,f(t)\,dt = \int_0^{t_w} h\,e^{-st}\,dt$$

$$F(s) = -\frac{h}{s}e^{-st}\Big|_0^{t_w} = \frac{h}{s}(1 - e^{-t_w s})$$

For a unit rectangular pulse h = 1 / t_w and the area under the pulse is unity.

IMPULSE FUNCTION

- A limiting case of the unit rectangular pulse is the **impulse** or **Dirac delta function**, which has the symbol δ(t). This function is obtained when $t_w \rightarrow 0$ while keeping the area under the pulse equal to unity; a pulse of infinite height and infinitesimal width results.

The δ(t) function is an example of a distribution or generalized function. It has the following properties:

- × $\delta(t) = 0$ if $t \neq 0$
- × $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
- × δ(0) is not defined
- × If g(t) is a continuous function on (-∞, ∞), then $\int_{-\infty}^{\infty} g(t)\,\delta(t)\,dt = g(0)$

Intuitively, we may think of δ(t) as an approximation of a physical transfer of one unit of charge at time zero. It can be shown that if *a* is a constant, then

- × $\delta(t-a) = 0$ if $t \neq a$
- × $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$
- × If g(t) is a continuous function on (-∞, ∞), then $\int_{-\infty}^{\infty} g(t)\,\delta(t-a)\,dt = g(a)$

From these formulas, we get that $\int_{-\infty}^{\infty} \delta(t)\,dt = \begin{cases} 0 & \text{if } t > 0 \\ 1 & \text{if } t < 0 \end{cases}$

Thus formally, $\int_{-\infty}^{t} \delta(t)\,dt = H(t)$

And δ(t) may be considered, in some sense, to be the derivative of the Heaviside function.

Proceeding formally, $\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$

In particular, $\mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\,\delta(t)\,dt = 1$

$$\mathcal{L}\{a\delta(t)\} = a$$