Example 6 (Atomic explosion)**.** *Supose that there is an atomic explosion. In such an explosion a lot of energy E is released instantaneoulsy in a point. A shockwave is then propagated from it. In this process, we assume that the radius r of the shockwave, the air density ρ, the time t and the energy E are the only dimensions that are involved in the law of how the shockwave propagates. Then, we have*

$$f(r, t, \rho, E) = 0.$$

Now that we have seen plenty of examples of laws, and seen that to all laws there is a function associated to it, could you think of a law that has no f related to it? It is hard to imagine it. Once we talk about relations between dimensions/units/quantities, equations appear. And, from each equation, we get a law!

Laws are important because they give as relations between the variables involved. If we know the law, then we know exactly their relation, but just knowing that there is a law tells us that there is some relation.

A **unit free law** is a law that does not depend on the choice of units. More concretely, given a law that depends on n quantities $q_1, \ldots, q_n$ and $m < n$ units $L_1, \ldots, L_m$,

$$f(q_1, \ldots, q_n) = 0,$$

and for any n $\lambda_i > 0$, the law is also true for the new variables $\hat{q}_i$ formed by the new units $\hat{L}_i = \lambda_i L_i$. That is,

$$f(\hat{q}_1, \ldots, \hat{q}_n) = 0.$$

Example 7. *An example of a unit free law is*

$$f(x, g, t) = x - \frac{1}{2}gt^2 = 0, \tag{2.1}$$

where x denotes position (L), g the constant of the gravitational field (L/T^2) and t time (T).

If $\hat{L} = \lambda_1 L$, $\hat{T} = \lambda_2 T$ then, since g has units in L/T^2, we get that

$$f(\hat{x}, \hat{g}, \hat{t}) = 0$$

if and only if Equation (2.1) *is also satisfied.*

2.3 Pi theorem.

Theorem 8. *Let*

$$f(q_1, \ldots, q_n) = 0$$

be a unit free physical law that relates the dimensioned quantities $q_1, \ldots, q_n$. Let $L_1, \ldots, L_m$ (where $m < n$) be the fundamental dimensions with

$$[q_i] = L_1^{a_{1,i}} \cdots L_m^{a_{m,i}}.$$

3.1 Regular perturbations.

The basic idea behind regular perturbations is the one behind Example 13: We do not need to perform any change in the equation and the Taylor expansion works fine.

Example 16. *Consider the initial value problem*

$$\begin{cases} mv' &= -av + bv^2 \\ v(0) &= V_0 \end{cases},$$

with $b \ll a$.

First, we introduce dimensionless variables

$$y = \frac{v}{V_0}, \quad \tau = \frac{at}{m},$$

obtaining the scaled initial value problem

$$\begin{cases} \dot{y} &= -y + \varepsilon y^2 \\ y(0) &= 1 \end{cases}, \tag{3.1}$$

where $\varepsilon = \dfrac{bV_0}{a} \ll 1$.

After this change of variables, the solution to Equation (3.1) when $\varepsilon = 0$ is

$$y_0(t) = e^{-t}.$$

Now, performing the ansatz that solutions to Equation (3.1) are of the form

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

and substituting it into Equation (3.1) we obtain

$$y_0'(t) + \varepsilon y_1'(t) + \varepsilon^2 y_2'(t) + h.o.t. = -y_0(t) - \varepsilon y_1(t) - \varepsilon^2 y_2(t) + \varepsilon \left(y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) \right)^2 + h.o.t.$$

which is equivalent to,

$$y_0'(t) + \varepsilon y_1'(t) + \varepsilon^2 y_2'(t) + h.o.t. = -y_0(t) - \varepsilon y_1(t) - \varepsilon^2 y_2(t) + \varepsilon y_0(t)^2 + \varepsilon^2 2y_0(t)y_1(t) + +h.o.t.$$

From this last equality we get

$$\begin{aligned} y_0(t) &= e^{-t}, \\ y_1(t) &= e^{-t} - e^{-2t}, \\ y_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}. \end{aligned}$$

Notice that in the definition of directional derivative we are using an auxiliary construction: a function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\varepsilon \to J(y_0 + \varepsilon v).$$

Exercise 35. *Compute the directional derivatives of the following functionals at the specified point y_0 and with direction v:*

1. $$J(y) = \int_0^1 y^2 dx,\ y_0 = \cos(x),\ v = \sin(x).$$

2. $$J(y) = \int_0^1 y'^2 dx,\ y_0 = \cos(x),\ v = \sin(x).$$

3. $$J(y) = \int_0^1 cos(y) dx,\ y_0 = x,\ v = x^2.$$

Now, with the help of directional derivatives we can give necessary conditions for the existence of minima/maxima of functionals.

Theorem 36. *Let $J : A \subset \mathcal{X} \to \mathbb{R}$ be a functional defined on an open subset of a normed vector space $\mathcal{X}$. If $y_0 \in A$ is a minimum (maximum) of J, then*

$$\delta J(y_0, v) = 0$$

for all v where the directional derivative exists.

Exercise 37. *Consider the functional $J : \mathcal{C}^0([2,4],\mathbb{R}) \to \mathbb{R}$,*

$$J(y) = \int_2^4 y(x)^2 dx.$$

Prove that $y_0(x) = 0$ is a minimum and check that

$$\delta J(0, v) = 0$$

for all $v \in \mathcal{C}^0([2,4]$.

4.3 The simplest problem.

The *simplest problem* in calculus of variations is to consider the functional

$$J(y) = \int_a^b L(x, y, y') dx \tag{4.1}$$

defined for functions $y \in \mathcal{C}^2[a,b]$ with the extra condition $y(a) = A$, $y(b) = B$. The function L should satisfy that it is twice differentiable in $[a,b] \times \mathbb{R}^2$.

When computing dirernctional derivatives it is required that $v(a) = v(b) = 0$, so $J(y + \varepsilon v)$ is well-defined.

4.4.2 Several functions.

Another way is by allowing several functions involved. For example, if two are involved, we get the functional

$$J(y) = \int_a^b L(x, y_1, y_1', y_2, y_2')dx,$$

with boundary conditions $y_1(a) = A_1, y_2(a) = A_2, y_1(b) = B_1, y_2(b) = B_2$. In this case, we get the system of equations

$$\begin{cases} \partial_{y_1} L - \dfrac{d}{dx}\partial_{y_1'} L = 0, \\ \partial_{y_2} L - \dfrac{d}{dx}\partial_{y_2'} L = 0. \end{cases}$$

4.4.3 Natural boundary conditions.

Another way of generalizing the Euler-Langrange equations is by allowing one of the boundaries free. For example, consider the functional

$$\int_a^b L(x, y, y')dx,$$

with boundary conditions $y(a) = A$ and $y(b)$ free. In this case, we get the system of equations

$$\begin{cases} \partial_y L - \dfrac{d}{dx}\partial_{y'} L = 0, \\ \partial_{y'} L(b, y(b), y'(b)) = 0. \end{cases}$$

4.5 More problems.

Exercise 44. *Find the extremal paths connecting two points lying on a sphere.*

Exercise 45. *Find the extremal paths connecting two points lying on a cylinder.*

Exercise 46. *Find the extremals of*

1. $$J(y) = \int_0^1 (y^2 + y'^2 - 2y\sin(x))dx,$$

   where $y(0) = 1$ *and* $y(1) = 2$.

2. $$J(y) = \int_1^2 \frac{y'^2}{x^3}dx,$$

   where $y(1) = 1$ *and* $y(2) = 0$.

3.

$$J(y) = \int_0^2 (y^2 + y'^2 + 2ye^x)dx,$$

where $y(0) = 0$ *and* $y(2) = 1$*.*

Exercise 47. *Find the Euler-Lagrange equation of the functional*

$$\int_a^b f(x)\sqrt{1+y'^2}dx,$$

and solve it for $y(a) = A$*,* $y(b) = B$*.*

Exercise 48. *Find an extremal for*

$$J(y) = \int_1^2 \frac{\sqrt{1+y'^2}}{x}dx,$$

where $y(1) = 0$ *and* $y(2) = 1$*.*

Exercise 49. *Show that the area of a surface given by the graph of a function* $z = f(x, y)$ *defined on a domain* D *is given by the double integral*

$$\iint_D \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\,dxdy.$$

It can be proved that a minimal surface satisfies the PDE

$$(1 + (\partial_x f)^2)\partial_{yy} f - 2\partial_x f \partial_y f \partial_{xy} f + (1 + (\partial_y f)^2)\partial_{xx} f = 0. \tag{4.4}$$

Prove that the surface given by $z = \arctan yx$ *satisfies Equation* (4.4)*.*
Could you give an idea of the proof of Equation (4.4)*?*

Exercise 50. *Find the extremals of*

1.

$$J(y) = \int_0^1 (y'^2 + y^2)dx,$$

where $y(0) = 1$ *and* $y(1)$ *free.*

2.

$$J(y) = \int_1^e \left(\frac{1}{2}x^2y'^2 - \frac{1}{8}y^2\right) dx,$$

where $y(1) = 1$ *and* $y(e)$ *free.*

Exercise 73. *Convince yourself that the discrete dynamical system defined on the unit interval* $[0,1]$

$$x_{n+1} = f(x_n),$$

where

$$f(x) = \begin{cases} 2x, & x < \frac{1}{2} \\ 2 - 2x, & x \geq \frac{1}{2} \end{cases}$$

is chaotic. That is, prove that it has a dense set of periodic orbits and its sensitive to initial conditions.

Exercise 74. *Convince yourself that the discrete dynamical system defined on the circle* $[0,1]/\mathbb{Z}$

$$x_{n+1} = f(x_n) \text{ (mod 1)},$$

where

$$f(x) = 2x \text{ (mod 1)}$$

is chaotic. That is, prove that it has a dense set of periodic orbits and its sensitive to initial conditions.

Where could we find this dynamical system?

We usually denote by capital letters the transformed function: $\mathcal{L}(y) = Y$.

Exercise 98. *Compute the Laplace transforms of the following functions:*

1. 1.
2. t.
3. t^n.
4. e^{at}.

The inverse of the Laplace transfrom is defined as

$$\mathcal{L}^{-1}(Y)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} Y(s)e^{st}dt,$$

where the integration path is a vertical line on the complex plane from bottom to top and a is chosen in a way that all singularities of the function Y lie on the left side of the vertical line with real part a.

The Laplace transform satisfies very nice properties.

Theorem 99. *The Laplace transform satisfies that*

$$\mathcal{L}(y^{(n)})(t) = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0).$$

Theorem 100. *Let's define the convolution of two functions $y_1, y_2 : [0, \infty) \to \mathbb{R}$ as*

$$(y_1 * y_2)(t) = \int_0^t y_1(t-s)y_2(s)ds.$$

Then, the Laplace transform satisfies that

$$\mathcal{L}(y_1 * y_2)(s) = Y_1(s)Y_2(s).$$

Furthermore,

$$\mathcal{L}^{-1}(Y_1Y_2)(t) = (y_1 * y_2)(t).$$

Exercise 101. *Prove Theorems 99 and 100.*

Exercise 102. *Prove that the Laplace transform defines a linear map. That is, $\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$, where a and b are constants.*

Have a look at Table 8.1, where some of the most common Laplace transforms appear.

Exercise 103. *With the help of Laplace transforms, solve the following problems:*

Observation 112. *Notice that both problems look similar. They only differ on the fact that for the Volterra equations the limits of integration depend on x, while for the Fredholm are fixed. As we will see, this small detail changes dramatically the way each problem is addressed.*

Let's discuss in more detail these equations. Notice that both equations can be written in the form

$$(K - \lambda Id)u = f, \tag{9.3}$$

where K denotes the linear integral operator. Hence, the equations will have a solution u if the function f is on the range of the linear operator $K - \lambda Id$. For example. if it is invertible:

$$u = (K - \lambda Id)^{-1} f.$$

Observation 113. *If the operator $K - \lambda Id$ fails to be invertible, it is still possible that for some (but not all) f Equation* (9.3) *has solutions.*

To study the invertibility of $K - \lambda Id$ it is important to understand for which λs the eigenvalue equation

$$Ku = \lambda u$$

is satisfied. For these, invertibility will fail.

The following exercise shows why studying the spectrum of a linear operator A is useful for solving linear systems.

Exercise 114. *Consider the real symmetric $n \times n$ matrix A. Give a solution of the nonhomogeneous system*

$$Au = \lambda u + f$$

in terms of the eigenvalues and eigenvectors of the matrix A. Use the fact that there exists an orthogonal basis of eigenvectors, and that the eigenvalues are all real.

9.1 Volterra equations.

As said before, Volterra equations are of the form

$$\int_a^x k(x,s)u(s)ds = \lambda u(x) + f(x),\ a \le x \le b. \tag{9.4}$$

There are special cases where the Volterra equation has an easy solution. Let's see some of these.

Exercise 115. *Suppose that the kernel k does not depend on the first variable x ($k(x,t) = g(t)$). Prove that a solution of Equation* (9.4) *satisfies the ODE*

$$u'(x) = \frac{1}{\lambda}(g(x)u(x) - f'(x)).$$

In this special case, the solution to the Fredholm equation (9.8) can be reduced to a finite dimensional linear algebra problem. Notice that it is equivalent to

$$\sum_{i=0}^{n} \alpha_i(x) \int_a^b \beta_i(y)u(y)dy - \lambda u(x) = f(x). \tag{9.9}$$

Let's denote by (f, g) the integrals

$$\int_a^b f(y)g(y)dy.$$

Multiplying Equation (9.9) by $\beta_j(x)$ and integrating with respect x we obtain the n linear equations of the form

$$\sum_{i=0}^{n} (\alpha_i, \beta_j)(\beta_i, u) - \lambda(\beta_j, u) = (\beta_j, f).$$

This system is of the form

$$Aw - \lambda w = b, \tag{9.10}$$

where A is the matrix with (i, j) entry (α_i, β_j), and w and f are vectors with entries (β_i, u) and (β_j, f_j).

If the linear system (9.10) has a solution w, then a solution to the Fredholm equation with degenerate kernel will be

$$u(x) = \frac{1}{\lambda}\left(-f(x) + \sum_{i=0}^{n} \alpha_i(x)w_i\right).$$

Observation 123. *Notice that the linear system* (9.10) *has a solution for all f if and only if λ is not an eigenvalue of the matrix A.*

It is easily proven in this case the following theorem, sometimes called the *Fredholm alternative.*

Theorem 124. *Consider the Fredholm equation* (9.8) *with degenerate kernel. Then, if λ is not an eigenvalue of the matrix A, the problem has a unique solution. If, on the contrary, it is an eigenvalue, either the problem has none or infinite number of solutions.*

Exercise 125. *Solve the Fredholm equation*

$$\int_0^1 xtu(t)dt + u(x) = \cos(2\pi x).$$

Exercise 126. *Solve the Fredholm equation*

$$\int_0^1 (xt + x^2t^2)u(t)dt + u(x) = \cos(2\pi x).$$

Appendix A

Solving some ODEs.

A.1 First order linear ODEs.

First order linear ODEs are of the form

$$y' + p(x)y = q(x). \tag{A.1}$$

First, we multiply Equation (A.1) by a function $f(x)$, obtaining

$$f(x)y' + f(x)p(x)y = f(x)q(x).$$

We will choose $f(x)$ such that

$$f'(x) = f(x)p(x) \tag{A.2}$$

Observation 1.30. *The solution to Equation* (A.2) *is*

$$f(x) = Ke^{\int p(x)dx}.$$

Thus, we get that

$$(f(x)y)' = f(x)p(x),$$

so

$$y(x) = \frac{1}{f(x)} \int f(x)p(x)dx.$$

A.2 Second order linear ODEs.

These are ODEs of the form

$$y'' + p(x)y' + q(x)y = r(x). \tag{A.3}$$

First, we find solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$