and $$\int_{-1}^{0} |x|\,dx = \int_{-1}^{0} (-x)dx = -\frac{1}{2}x^2\Big|_{-1}^{0} = 0 + \frac{1}{2} = \frac{1}{2}$$

Thus, $$\int_{0}^{1} |x|\,dx = \int_{0}^{1} x dx = \frac{1}{2}x^2\Big|_{0}^{1} = \frac{1}{2} - 0 = \frac{1}{2}.$$

$$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) Since $|x-3| = -x+3$ for $x \le 3$ and $|x-3| = x-3$ for $x \ge 3$, you get

$$\begin{aligned}\int_0^4 (1+|x-3|)^2 dx &= \int_0^3 [1+(-x+3)]^2 dx + \int_3^4 [1+(x-3)]^2 dx = \\ &= \int_0^3 (-x+4)^2 dx + \int_3^4 (x-2)^2 dx = \\ &= -\frac{1}{3}(-x+4)^3\Big|_0^3 + \frac{1}{3}(x-2)^3\Big|_3^4 = \\ &= -\frac{1}{3} + \frac{64}{3} + \frac{8}{3} - \frac{1}{3} = \frac{70}{3}.\end{aligned}$$

9. (a) Show that if F is an antiderivative of f, then

b

$f(-x)dx = -F(-b) + F(-a)$

a

(b) A function f is said to be even if $f(-x) = f(x)$. [For example, f $f(x) = x2$ is even.] Use problem 8 and part (a) to show that if is even, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

(c) Use part (b) to evaluate $\int_{-1}^{1} |x|\,dx$ and $\int_{-2}^{2} x^2 dx$.

(d) A functionand part (a) to show that if f is said to be odd if $f(-x) = -f(x)$.

Use problem 8 f is odd, then

. $$\int_{-a}^{a} f(x)dx = 0$$

$\int_{12}^{12} x^3 dx.$

(e) EvaluateandSolution.$u(b)$ =⁻(−a) Substitute$b$. Hence, $u = -x$. Then du

$= -dx$, $u(a) = -a$

6.6 Area and Integration

In problems, 1 through 9 find the area of the region R.

1. *resolution*.is the triangle with vertices from the corresponding graph (Figure 6.1) you see that the$(-4,0)$, $(2,0,)$ and $(2,6)$. region in question is bellow the line $y = x + 4$ above the *x-axis*, and extends from $x = -4$ to $x = 2$.

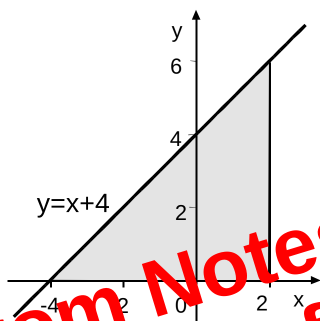

Figure 6.1

$$A = \int_{-4}^{2} (x+4)dx = \left(\frac{1}{2}x^2 + 4x\right)\Big|_{-4}^{2} = (2+8) - (8 -$$

R is the region bounded by the curve $y = e^x$, the lines $x = 0$

Hence,

16) = 18.

2.and $x = \ln \frac{1}{2}$, and the x axis.

responding graph (Figure 6.2) you see that the region in question

isSolution. Since $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$ ' -0.7, from the corbellow the line

$y = e^x$ above the x axis, and extends from $x = \ln \frac{1}{2}$ to $x = 0$.

$$A_2 = \int_2^{10} (-x+10)dx = \left(-\frac{1}{2}x^2 + 10x\right)\Big|_2^{10} = -50+100+2-20 = 32$$

Therefore,

$$A = A_1 + A_2 = \frac{32}{3} + 32 = \frac{128}{3}.$$

4. R is the region bounded by the curves $y = x^2 + 5$ and $y = -x^2$, the line $x = 3$, and the y axis.

Solution. Sketch the region as shown in Figure 6.4.

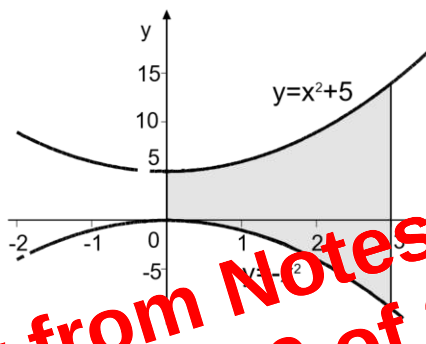

Figure 6.4.

Notice that the region in question is bounded above by the curve $y = x^2 + 5$ and below by the curve $y = -x^2$ and extends from $x = 0$ to $x = 3$. Hence,

$$A = \int_0^3 [(x^2+5)-(-x^2)]dx = \int_0^3 (2x^2+5)dx = \left(\frac{2}{3}x^3 + 5x\right)\Big|_0^3 = 18+15 = 33.$$

R is the region bounded by the curves $y = x^2 - 2x$ and $y = -x^2 + 4$

5..

Solution. First make a sketch of the region as shown in Figure 6.5 and find the points of intersection of the two curves by solving the equation

$$x^2 - 2x = -x^2 + 4 \qquad \text{i.e.} \qquad 2x^2 - 2x - 4 = 0$$

to get

$$x = -1 \qquad \text{and} \qquad x = 2.$$

The corresponding points $(-1,3)$ and $(2,0)$ are the points of intersection.

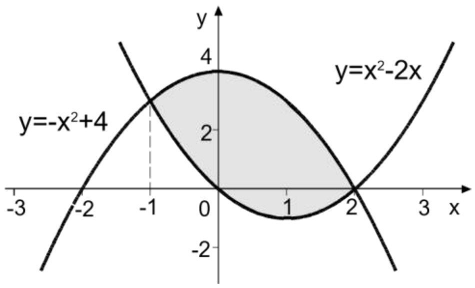

Figure 6.5.

Notice that for $-1 \le x \le 2$, the graph of $y = -x^2 + 4$ lies above that of $y = x^2$ $- 2x$. Hence,

$$\begin{aligned} A &= \int_{-1}^{2} [(-x^2+4) - (x^2-2x)]dx = \int_{-1}^{2} (-2x^2+2x+4)dx = \\ &= \left(-\frac{2}{3}x^3 + x^2 + 4x\right)\Big|_{-1}^{2} = -\frac{16}{3} + 4 + 8 - \frac{2}{3} - 1 + 4 = 9. \end{aligned}$$

6. R is the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

Solution. Sketch the region as shown in Figure 6.6. Find the points of intersection by solving the equations of the two curves simultaneously to get

$$x^2 = \sqrt{x} \qquad x^2 - \sqrt{x} = 0 \qquad \sqrt{x}(x^{\frac{3}{2}} - 1) = 0$$

$- x = 0$ and $x = 1$.

The corresponding points $(0,0)$ and $(1,1)$ are the points of intersection.

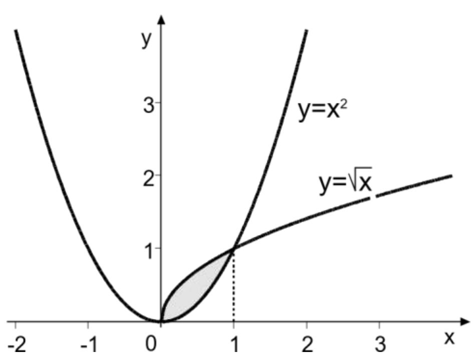

Figure 6.6.

Notice that for2 $0 \le x \le 1$, the graph lies above that of $y = x$. Hence,

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

(a) R is the region to the right of the y axis that is bounded above by the curve $y = 4 - x^2$ and below the line $y = 3$.

(b) R is the region to the right of the y axis that lies below the line $y = 3$ and is bounded by the curve $y = 4 - x^2$, the line $y = 3$, and the coordinate axes.

Solution. Note that the curve $y = 4 - x^2$ and the line $y = 3$ intersect to the right of the y axis at the point (1,3), since x = 1 is the positive solution of the equation(a) Sketch the region as shown in Figure4 −6.7.$x^2 = 3$, i.e. $x^2 = 1$.

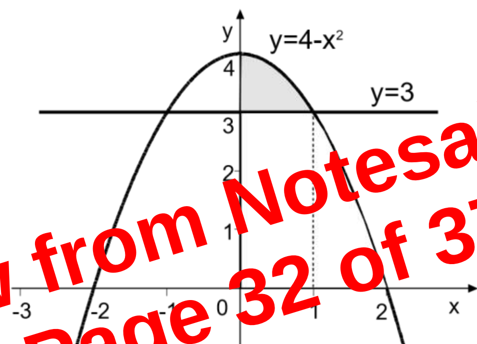

Figure 6.7.

ofNotice that for $0 \le x \le 1$, the graph of $y = 4 - x^2$ lies above that

$y = 3$. Hence,

$$A = \int_0^1 (4-x^2-3)dx = \int_0^1 (1-x^2)dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 = 1-\frac{1}{3} = \frac{2}{3}.$$

(b) Sketch the region as shown in Figure 6.8.

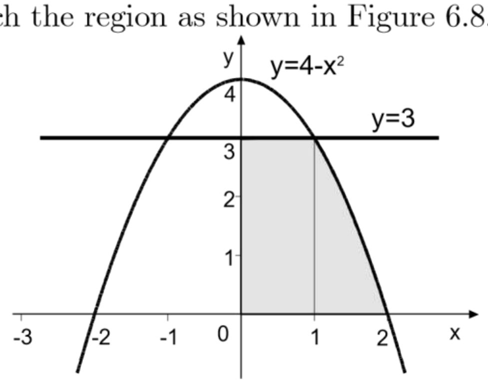

Figure 6.8.

Then break R into two subregions, R_1 that extends from $x = 0$ to $x = 1$ and R_2 that extends from $x = 1$ to $x = 2$, as shown in Figure 6.9. Hence, the area of the region R_1 is

$$A_1 = \int_0^1 \left(x - \frac{x}{8}\right) dx = \int_0^1 \frac{7}{8} x dx = \frac{7}{16} x^2 \Big|_0^1 = \frac{7}{16}$$

and the area of the region R_2 is

$$A_2 = \int_1^2 \left(\frac{1}{x^2} - \frac{x}{8}\right) dx = \left(-\frac{1}{x} - \frac{1}{16} x^2\right)\Big|_1^2 = -\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16} = \frac{5}{16}.$$

Thus, the area of the region R is the sum

$$A = A_1 + A_2 = \frac{12}{16} = \frac{3}{4}.$$

8. R is the region bounded by the curves $y = x^3 - 2x^2 + 5$ and $y = x$Solution^2 + $4x - 7$. First, make a rough sketch of the two curves as shown in Figure 6.10. You find the points of intersection by solving the equations of the two curves simultaneously

$$x^3 - 2x^2 + 5 = x^2 + 4x - 7 \qquad x^3 - 3x^2 - 4x + 12 = 0$$
$$x^2(x-3) - 4(x-3) = 0 \qquad (x-3)(x-2)(x+2) = 0$$

to get $x = -2$, $x = 2$ and $x = 3$.

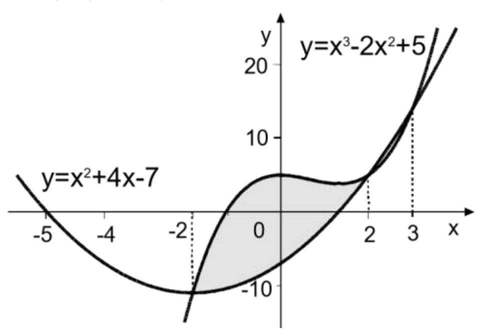

Figure 6.10.

The region whose area you wish to compute lies between $x = -2$ and $x = 3$, but since the two curves cross at $x = 2$, neither curve is always above the other between $x = -x2_2$ +4andxx−== 37$x$between3. However, since the curve− 2x2 + 5x =between−2$x$and= −xx2= 2= 2and, $y = x^3 - 2x^2 + 5$ is above