Answer: Note that the domain of f i.e. the set D is a closed and bounded subset of $\mathbb{R}^2$ and hence it is a compact subset of $\mathbb{R}^2$. Also note that D is path-connected (if we take any two points in D then there always exist a path between them which lies entirely in D) and hence it is connected. (Try to draw the picture of D to visualize easily the compactness and connectedness).

Now as f is continuous and D is compact therefore the range of f i.e. $f(D)$ must be a compact subset of $\mathbb{R}$. Therefore $f(D)$ cannot be an open interval. Hence the option (c) cannot be true. Again as f is continuous and D is connected therefore the range of f i.e. $f(D)$ must be a connected subset of $\mathbb{R}$. Hence option (c) is excluded because any finite subset of $\mathbb{R}$ which contain more than one element is not connected. Now suppose that f is a constant function over D. Then $f(D)$ is a singleton set. Likewise if f is a non-constant continuous function then $f(D)$ must be a compact and connected subset of $\mathbb{R}$. i.e. $f(D)$ must be a closed and bounded interval. Therefore options (a) and (d) may be correct.

8. Question: Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers such that $na_n < b_n < n^2 a_n$. The radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ is 4, then the power series $\sum\limits_{n=0}^{\infty} b_n x^n$

a) Converges for all x with $|x| < 2$.

b) Converges for all x with $|x| > 2$.

c) does not converge for any x with $|x| > 2$.

d) does not converge for any x with $|x| < 2$.

Answer : If $x = 0$, then $\sum\limits_{n=0}^{\infty} b_n x^n = 0$. So option (d) is false. Now let us find a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence 4. Well notice that radius of convergence of $\sum\limits_{n=0}^{\infty} x^n$ is 1. Hence radius of convergence of $\sum\limits_{n=0}^{\infty} (\frac{x}{4})^n$ is 4, i.e. $a_n = \frac{1}{4^n}$ is a good choice. Now

$$n < n+1 < n^2 \implies \frac{n}{4^n} < \frac{n+1}{4^n} < \frac{n^2}{4^n}$$

for all $n \geq 2$. Therefore if we take $b_n = \frac{n+1}{4^n}$, then this b_n satisfies all the given conditions. So consider the power series $\sum\limits_{n=0}^{\infty} \frac{n+1}{4^n} x^n$. Now what if $x = 4$ and $x = 3$. Can you see that $\sum\limits_{n=0}^{\infty} \frac{n+1}{4^n} x^n$ diverges for $x = 4$ and converges for $x = 3$(apply Ratio test). Therefore the options (b) and (c) are false. We have eliminated all of (b), (c) and (d). So option (a) correct (Nice! Still want to verify option (a). Try to prove it. Hint: $\frac{n+1}{4^n} < \frac{n^2}{4^n} = (\frac{n}{2^n})^2 = (c_n)^2$, then compare the radius of convergence of $\sum\limits_{n=0}^{\infty} b_n x^n$ and $\sum\limits_{n=0}^{\infty} c_n x^n$).

9. Question The set $A = \{\frac{x}{1+x} : -1 < x < 1\}$, as a subset of $\mathbb{R}$. Then A is

a) compact and connected.

b) compact and not connected.

c) not compact and connected.

d) not compact and not connected.

Answer: Consider the function $f : (-1, 1) \to \mathbb{R}$ given by,

$$y = f(x) = \frac{x}{1+x}$$

Then this is a continuous function. $(-1, 1)$ is a connected set in $\mathbb{R}$. So The given set A is connected as $A = f(-1, 1)$. Again as $x \to 1-$, $y \to \frac{1}{2}$. Therefore $\frac{1}{2}$ is a limit point of A, which does not belong to A. So A is not closed and hence A is not compact in $\mathbb{R}$ (can you see that A is unbounded also). Therefore the option (c) is the correct answer.