Example 5:

A truck driver can take three different routes in order to get from city A to city B, then one of four routes from city B to city C, and finally one of three routes from city C to city D. How many possible routes are there if the driver must go from city A to B to C and then to D?

$3 \times 4 \times 3 = 36$ different routes from A to D.

1.4.2 Permutations and Combinations

Permutations: Ordered set of elements also called orderings.

The total number of orderings of n elements is $n!$ (called the permutation of n elements).

The following question arises: In how many ways can r elements be chosen from a set C with n elements? There are two situations:

i) Sampling with replacement: Each time an element is chosen it is chosen from the entire set C (an element can be chosen more than once). All the elements have the same probability of being chosen at any point in time. This is done in $n \times n \times n \times n \ldots \times n = n^r$ many ways (multiplication principle).

ii) Sampling without replacement: Each time an element is chosen it is removed from set C (an element cannot be chosen more than once). Each of the remaining elements has the same probability of being chosen. This is done in $n \times (n-1) \times (n-2) \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$ many ways.

Example 6

Suppose you have three books A, B and C, but you have room form only two on your bookshelf. In how many ways can you select and arrange the two books?

This is an example of sampling without replacement, therefore the number of ways to arrange the two books is $(n-1) = 3(2) = 6\ \left[\frac{n!}{(n-r)!} = 6\right]$.

Example 7

A "word" is an ordered collection of letters. In English, how many four letter words are possible? Supposing each of the 26 letters in the alphabet are chosen (with replacement) to be used, there will be $(26)(26)(26)(26) = n^4 = 456{,}976$ many possible four letter words that could be formed.

Combinations: An unordered set of elements. We are no longer interested in ordered samples just in the samples regardless of the order.

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) = \sum_{i=1}^{3} P(A|B_i)P(B_i)$$

Example 12

A statistics student is randomly chosen at the university. This student is either a first, second or third year student with probability 0.5, 0.3 and 0.2 respectively. The probability that the student will pass statistics in the semester is respectively 0.4, 0.6 and 0.8, for the three years. Calculate the probability that the student will pass.

Let A be the event that the student pass and B_1 the event that the student is a first year, B_2 that the student is a second year, and B_3 that the student is a third year. Then

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$
$$= 0.4(0.5) + 0.6(0.3) + 0.8(0.2) = 0.54$$

Bayes'rule: Let $B_1, B_2, \ldots, B_n$ be events such that $\bigcup_{i=1}^{n} B_i = \Omega$ and $B_i \cap B_j = \emptyset$ (disjoint) for $i \neq j$ with $P(B_i) > 0$ for all i. Then

$$P(B_j|A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Example 13 (Continuation of Example 12)

Calculate the probability that a randomly chosen Statistics student is a second year student, given that he has failed the semester.

$$P(B_2|A^c) = \frac{P(A^c|B_2)P(B_2)}{P(A^c)} = \frac{[1 - P(A|B_2)]P(B_2)}{1 - P(A)} = \frac{(1 - 0.6)0.3}{1 - 0.54} = 0.2609$$

NOTE: $P(A^c|B) = 1 - P(A|B)$, but $P(A|B^c) \neq 1 - P(A|B)$

Example

Let BC denote the event Breast Cancer and + the event of a positive mammogram.

$$P(BC) = 0.01, P(\overline{BC}) = 0.99, P(+|BC) = 0.8, P(+|\overline{BC}) = 0.1$$

$$P(BC|+) = \frac{P(+|BC)P(BC)}{P(+|BC)P(BC) + P(+|\overline{BC})P(\overline{BC})} = \frac{0.8(0.01)}{0.8(0.01) + 0.1(0.99)} = \frac{0.008}{0.107}$$
$$= 0.0748 = 7.5\%$$