The values (mass points) that the random variable can take on is $x = \{0,1,2,3\}$.

The probability mass function is then:

$$p(0) = P(X = 0) = \frac{1}{8}$$

$$p(1) = P(X = 1) = \frac{3}{8}$$

$$p(2) = P(X = 2) = \frac{3}{8}$$

$$p(3) = P(X = 3) = \frac{1}{8}$$

ii) Let the random variable Y denote the number of tails.

The values (mass points) that the random variable can take on is $y = \{0,1,2,3\}$.

The probability mass function is then:

$$p(0) = P(Y = 0) = \frac{1}{8}$$

$$p(1) = P(Y = 1) = \frac{3}{8}$$

$$p(2) = P(Y = 2) = \frac{3}{8}$$

$$p(3) = P(Y = 3) = \frac{1}{8}$$

iii) Let the random variable Z denote the number of heads minus the number of tails.

The values (mass points) that the random variable can take on is $z = \{-3,-1,1,3\}$.

The probability mass function is then:

$$p(-3) = P(Z = -3) = \frac{1}{8}$$

$$p(-1) = P(Z = -1) = \frac{3}{8}$$

$$p(1) = P(Z = 1) = \frac{3}{8}$$

$$p(3) = P(Z = 3) = \frac{1}{8}$$

2.1.5 Poisson random variable

Poisson probability mass function: $p(k) = P(X = k) = \begin{cases} e^{-\lambda}\frac{\lambda^k}{k!} & \text{for k=0,1,2,...} \\ 0 & \text{otherwise} \end{cases}$ with parameter $\lambda > 0$.

Properties of a Poisson mass function:

i) $0 \leq p(k) \leq 1$

ii) $\sum_{k=1}^{\infty} p(k) = 1$

Proof:

$$\sum_{k=0}^{\infty} p(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\left\{1 + \lambda^1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right\} = e^{-\lambda}e^{\lambda} = 1$$

Example 19

The average number of traffic accidents on the N12 between Potchefstroom and Johannesburg is 2 per week. Assume that the number of accidents in a week follow a Poisson distribution with $\lambda = 2$.

i) Calculate the probability that no accidents occur between Potchefstroom and Johannesburg during a 1-week period.

The average number of accident per week is $\lambda = 2$. Let X be the number of accidents in a 1 week period.

$$P(X = 0) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-2}\frac{2^0}{0!} = 0.135335$$

ii) Calculate the probability that at most three accidents occur during a two week period.

During a two week period the average number of accidents is equal to $\lambda = 4$.

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= e^{-4}\frac{4^0}{0!} + e^{-4}\frac{4^1}{1!} + e^{-4}\frac{4^2}{2!} + e^{-4}\frac{4^3}{3!} = 0.433471$$

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$$

Cumulative distribution function: The cdf of a continuous random variable is defined just as it was in the discrete case:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$$

There exists a one-to-one relationship between distribution functions and density funtions:

i) $F(x) = \int_{-\infty}^{x} f(u)du$ (by definition)

ii) $F'(x) = f(x)$

There is also a relationship between distribution functions and probabilities:

$$P(a \le X \le b) = \int_{a}^{b} f(x)dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)dx = P(X \le b) - P(X \le a)$$
$$= F(b) - F(a)$$

Example 22

If X has the probability density

$$f(x) = ke^{-3x}, \quad x > 0$$

Find k and $P\left(\frac{1}{2} \le X \le 1\right)$.

i) $\int_{0}^{\infty} ke^{-3x}dx = 1$

$$k\left[\frac{1}{-3}e^{-3x}\right]_0^{\infty} = 0 - \left(-\frac{k}{3}\right) = \frac{k}{3} = 1$$

Therefore $k = 3$.

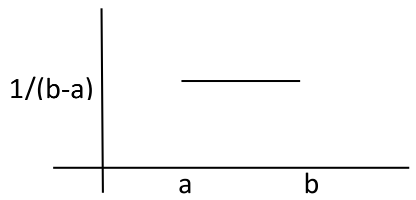
ii) $P\left(\frac{1}{2} \le X < 1\right) = \int_{\frac{1}{2}}^{1} 3e^{-3x}dx = 3\left[-\frac{1}{3}e^{-3x}\right]_{\frac{1}{2}}^{1} = -e^{-3} - \left(-e^{-\frac{3}{2}}\right) = 0.173$

Example 23

Find the cdf of the random variable X in the previous example, and use it to re-evaluate $P\left(\frac{1}{2} \le X \le 1\right)$.

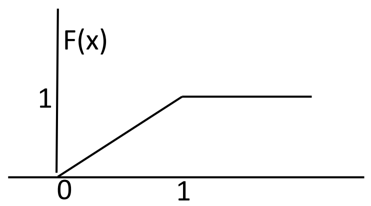
i) $F(x) = \int_{-\infty}^{x} f(u)du = \int_{0}^{x} 3e^{-3u}du = 3\left[-\frac{1}{3}e^{-3u}\right]_0^{x} = -e^{-3x} - (-e^{0}) = 1 - e^{-3x}$, $x > 0$

ii) $P\left(\frac{1}{2} \le X < 1\right) = F(1) - F\left(\frac{1}{2}\right) = (1 - e^{-3}) - \left(1 - e^{-\frac{3}{2}}\right) = 0.173$



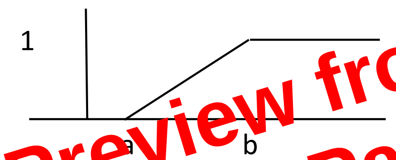
The cdf of a uniform random variable on [0, 1] is:

$$F(x) = \int_0^x f(u)du = \int_0^x 1du = [u]_0^x = x, \text{ for } 0 \leq x \leq 1$$



The cdf of a uniform random variable on [a, b] is:

$$F(x) = \int_a^x f(u)du = \int_a^x \frac{1}{b-a}du = \frac{1}{b-a}[u]_a^x = \frac{x-a}{b-a}, \text{ for } a \leq x \leq b$$



Example 28

At a particular hospital a study is done over a long period of time (covering many patients) and it is found that the distribution of “time of death” is uniform over the course of a day for a random unknown patient.

i) What is the probability that a random patient dies before 8am on the day of their death?

$X \sim U[0,24]$

$$P(X < 8) = F(8) = \frac{8-0}{24-0} = 0.3333$$

ii) What is the probability that a random patient dies during visiting hours, if visiting hours are 10am to noon?