Evaluate the Integral: $\mathbf{I} = \int_0^{+\infty} e^{-\sqrt{x}} \ln\left(1 + \frac{1}{\sqrt{x}}\right) dx$

$I = \int_0^{+\infty} e^{-\sqrt{x}} \ln\left(1 + \frac{1}{\sqrt{x}}\right) dx$, let $t = \sqrt{x}$, then $dt = \frac{1}{2\sqrt{x}} dx = \frac{1}{2t} dx$, so $dx = 2t dt$, then:

$I = 2\int_0^{+\infty} te^{-t} \ln\left(1 + \frac{1}{t}\right) dt = 2\int_0^{+\infty} te^{-t} \ln\left(\frac{t+1}{t}\right) dt$, then we get:

$I = 2\int_0^{+\infty} te^{-t} \ln(1+t)\, dt - 2\int_0^{+\infty} te^{-t} \ln t\, dt = 2J - 2\int_0^{+\infty} te^{-t} \ln t\, dt$

Now evaluating: $J = \int_0^{+\infty} te^{-t} \ln(1+t)\, dt$, using integration by parts:

Let $u = t\ln(1+t)$, then $u' = \ln(1+t) + \frac{t}{1+t}$ and let $v' = e^{-t}$, then $v = -e^{-t}$, then we get:

$J = [-te^{-t}\ln(1+t)]_0^{+\infty} + \int_0^{+\infty} \left[\ln(1+t) + \frac{t}{1+t}\right] e^{-t} dt$

$J = \int_0^{+\infty} e^{-t} \ln(1+t)\, dt + \int_0^{+\infty} \frac{te^{-t}}{1+t} dt$

Note that: $\int_0^{+\infty} \frac{te^{-t}}{1+t} dt = \int_0^{+\infty} \frac{(t+1-1)e^{-t}}{1+t} dt = \int_0^{+\infty} e^{-t} dt - \int_0^{+\infty} \frac{e^{-t}}{1+t} dt = 1 - \int_0^{+\infty} \frac{e^{-t}}{1+t} dt$

Now integrating $\int_0^{+\infty} \frac{e^{-t}}{1+t} dt$ **by parts:** let $u = e^{-t}$, then $u' = -e^{-t}$ and let $v' = \frac{1}{1+t}$, then

$v = \ln(1+t)$, then we get: $\int_0^{+\infty} \frac{e^{-t}}{1+t} dt = [e^{-t}\ln(1+t)]_0^{+\infty} + \int_0^{+\infty} e^{-t} \ln(1+t)\, dt$

Then: $\int_0^{+\infty} \frac{e^{-t}}{1+t} dt = \int_0^{+\infty} e^{-t} \ln(1+t)\, dt$, then:

$J = \int_0^{+\infty} e^{-t} \ln(1+t)\, dt + \int_0^{+\infty} \frac{te^{-t}}{1+t} dt = \int_0^{+\infty} e^{-t} \ln(1+t)\, dt + 1 - \int_0^{+\infty} \frac{e^{-t}}{1+t} dt$

$J = \int_0^{+\infty} e^{-t} \ln(1+t)\, dt + 1 - \int_0^{+\infty} e^{-t} \ln(1+t)\, dt = 1$

So we get: $I = 2 - 2\int_0^{+\infty} te^{-t} \ln t\, dt = 2\left(1 - \int_0^{+\infty} te^{-t} \ln t\, dt\right)$

Now let's evaluate the integral: $\int_0^{+\infty} te^{-t} \ln t\, dt$

Definition of gamma function: $\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$ and $\Gamma'(s) = \int_0^{+\infty} t^{s-1} e^{-t} \ln t\, dt$

Definition of digamma function: $\Psi(s) = \frac{s}{ds} \ln(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)}$, so $\Gamma'(s) = \Gamma(s)\,\Psi(s)$, then we can write: $\Gamma(s)$

$\Psi(s) = \int_0^{+\infty} t^{s-1} e^{-t} \ln t\, dt$, substituting $s = 2$, we get $\Gamma(2)\,\Psi(2) = \int_0^{+\infty} te^{-t} \ln t\, dt$

We have $\Gamma(2) = 1! = 1$ and $\Psi(s+1) = \Psi(s) + \frac{1}{2}$, so for $s = 1$ $\Psi(2) = \Psi(1) + 1$, but we have

$\Psi(1) = -\gamma$, then we get: $\Psi(2) = 1 - \gamma$, then $\int_0^{+\infty} te^{-t} \ln t\, dt = \Gamma(2)\,\Psi(2) = 1 - \gamma$

Therefore: $I = 2[1 - (1 - \gamma)] = \mathbf{2\,\gamma}$