Evaluate the integral: $$I=\int_0^1 \ln^{2K}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx$$

Let $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$, $x=0\Rightarrow\theta=0$ & $x=1\Rightarrow\theta=\pi/2$

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)=\ln\left(\frac{1-\sqrt{1-\sin^2\theta}}{\sin\theta}\right)=\ln\left(\frac{1-\cos\theta}{\sin\theta}\right);\text{with } 0<\theta<\frac{\pi}{2}$$

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)=\ln\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)=\ln\left(\frac{\sin\frac{\theta}{2}}{\cos\theta/2}\right)=\ln\left(\tan\frac{\theta}{2}\right)$$

Similarly; we have: $\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)=\ln\left(\cot\frac{\theta}{2}\right)$; then:

$$\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}=\frac{\ln\tan\frac{\theta}{2}}{\ln\cot\frac{\theta}{2}}=\frac{\ln\tan\frac{\theta}{2}}{\ln\left(1/\tan\theta/2\right)}=\frac{\ln\tan\theta/2}{-\ln\tan\theta/2}=-1\Longrightarrow$$

$$I=\int_0^1 \ln^{2K}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx=\int_0^{\pi/2}\ln^{2K}(-1)\times\cos\theta\, d\theta$$

But $\ln z=\ln|z|+i\arg z$; so $\ln(-1)=\ln|-1|+i\pi=i\pi\Longrightarrow$

$$\ln^{2K}(-1)=(i\pi)^{2K}=i^{2K}.\pi^{2K}=(i^2)^K.\pi^{2K}=(-1)^K.\pi^{2K}\Longrightarrow$$

$$I=(-1)^K.\pi^{2K}\int_0^{\pi/2}\cos\theta\, d\theta=(-1)^K.\pi^{2K}(1-0)=(-1)^K.\pi^{2K}$$

$$\therefore\int_0^1 \ln^{2K}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx=(-1)^K.\pi^{2K}$$