Evaluate the integral: $I = \int_0^1 \frac{\Psi\left(x+\frac{3}{2}\right) - \Psi\left(x-\frac{1}{2}\right)}{x} dx$

We have $\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \Gamma\left(1+\left(\frac{1}{2}+x\right)\right)\Gamma\left(1+\left(\frac{1}{2}-x\right)\right)$

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}-x\right)$, then we get:

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)\left[\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)\right]$

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{4}-x^2\right)\left[\frac{\pi}{\cos(\pi x)}\right] = \left(\frac{1}{4}-x^2\right)\pi\sec(\pi x)$, apply ln to both sides, we get:

$\ln\Gamma\left(\frac{3}{2}+x\right) + \ln\Gamma\left(\frac{3}{2}-x\right) = \ln\left(\frac{1}{4}-x^2\right) + \ln\pi + \ln\sec(\pi x)$, derive both sides w.r.t x, we get:

$$\frac{\Gamma'\left(\frac{3}{2}+x\right)}{\Gamma\left(\frac{3}{2}+x\right)} - \frac{\Gamma'\left(\frac{3}{2}-x\right)}{\Gamma\left(\frac{3}{2}-x\right)} = -\frac{2x}{\frac{1}{4}-x^2} + \frac{\pi\sec(\pi x)\tan(\pi x)}{\sec(\pi x)};\quad \text{then we get:}$$

$$\Psi\left(\frac{3}{2}+x\right) - \Psi\left(\frac{3}{2}-x\right) = \pi\tan(\pi\pi) - \frac{8x}{1-4x^2} \;\ldots(1)$$

We have $\ln\Gamma(x) + \ln\Gamma(1-x) = \ln\pi - \ln\sin(\pi x)$, by deriving both sides w.r.t x, we get:

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1-x)}{\Gamma(1-x)} = -\pi\frac{\cos(\pi x)}{\sin(\pi x)} \Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1-x)}{\Gamma(1-x)} = -\pi\cot(\pi x)$$

With $\frac{\Gamma'(x)}{\Gamma(x)} = \Psi(x)$ and $\frac{\Gamma'(1-x)}{\Gamma(1-x)} = \Psi(1-x)$; so $\Psi(x) - \Psi(1-x) = -\pi\cot(\pi x)$

So replacing x by $x-\frac{1}{2}$, we get:

$$\Psi\left(x-\frac{1}{2}\right) - \Psi\left(1-\left(x-\frac{1}{2}\right)\right) = \Psi\left(x-\frac{1}{2}\right) - \Psi\left(\frac{3}{2}-x\right) = -\pi\cot\left(\pi\left(x-\frac{1}{2}\right)\right);\quad \text{then}$$

$$\Psi\left(\frac{3}{2}-x\right) - \Psi\left(x-\frac{1}{2}\right) = \pi\frac{\cos\left(\pi x-\frac{\pi}{2}\right)}{\sin\left(\pi x-\frac{\pi}{2}\right)} = \frac{\pi\sin(\pi x)}{-\pi\cos(\pi x)} = -\pi\tan(\pi x) \;\ldots(2)$$

then we get: $\Psi\left(x-\frac{1}{2}\right) - \Psi\left(\frac{3}{2}-x\right) = \pi\tan(\pi x) \;\ldots(2)$; subtract (1) & (2) we get:

$$\Psi\left(x-\frac{1}{2}\right) - \Psi\left(\frac{3}{2}-x\right) - \Psi\left(\frac{3}{2}+x\right) + \Psi\left(\frac{3}{2}-x\right) = \pi\tan(\pi x) - \pi\tan(\pi\pi) + \frac{8x}{1-4x^2}$$

Therefore ; we get: $\Psi\left(x-\frac{1}{2}\right) - \Psi\left(x+\frac{3}{2}\right) = \frac{8x}{1-4x^2}$; so $\Psi\left(x+\frac{3}{2}\right) - \Psi\left(x-\frac{1}{2}\right) = \frac{8x}{4x^2-1}$

$$I = \int_0^1 \frac{1}{x}\cdot\frac{8x}{4x^2-1}dx = 4\int_0^1 \frac{2}{4x^2-1}dx = 4\int_0^1 \frac{2x+1-2x+1}{(2x+1)(2x-1)}dx;\ \text{then}$$

$$I = 4\int_0^1 \frac{(2x+1)-(2x-1)}{(2x+1)(2x-1)}dx = 4\int_0^1 \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)dx = 2\int_0^1 \left(\frac{2}{2x-1} - \frac{2}{2x+1}\right)dx$$

$$I = 2\left[\ln\left|\frac{2x-1}{2x+1}\right|\right]_0^1 = 2\ln\frac{1}{3} - 2\ln 1 = \ln\left(\frac{1}{3}\right)^2 = \ln\left(\frac{1}{9}\right)$$