1.1.4 Subsequences and the Bolzano-Weiertrass Theorem

Definition 1.5. Let $\{a_n\}$ be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3} \cdots$$

is called a subsequence of $\{a_n\}$ and is denoted by $\{a_{n_k}\}$, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 1.6. *If $\{a_n\}$ is a sequence converging to L, then every subsequence $\{a_{n_k}\}$ also converges to L.*

Theorem 1.7. *Let $\{a_n\}$ be a sequence. The $\{a_n\}$ has a monotone subsequence.*

Proof. To prove this theorem, we call the k-th term *dominant* if it is greater than or equal to all the following terms. In other words, the term a_k is dominant if $a_k \geq a_m$ for all $m \geq k$. There are two cases to consider:

Case 1: There are infinitely many dominant terms in the sequence $\{a_n\}$. Then we have a infinite subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ Now a_{n_i} is dominant for all $i \in \mathbb{N}$ and $n_i < n_{i+1}$, so we must have$a_{n_i} \geq a_{n_{i+1}}$. Similarly, we have

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \cdots$$

Hence we have a decreasing subsequence a_{n_k} and we are done.

Case 2: There are only finitely many dominant terms in the sequence $\{a_n\}$. (including the case where there is no dominant term). Suppose a_N is the last dominant term, and let $n_1 = N + 1$. Now a_{n_1} is not dominant and so there must exist $n_2 > n_1$ such that $a_{n_1} \leq a_{n_2}$. Continuing this way, we obtain an increasing subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \cdots$ as desired. $\square$

Theorem 1.8 (the Bolzano-Weiertrass Theorem). *Let $\{a_n\}$ be a bounded sequence. Then $\{a_n\}$ has a convergent subsequence.*

limit function? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f ?

To say that f is continuous at a limit point x means

$$\lim_{t \to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t),$$

i.e., whether the order in which limit processes are carried out is immaterial.

We shall now show by means of several examples that limit processes cannot in general be interchanged without affecting the result. Afterward, we shall prove that under certain conditions the order in which limit operations are carried out is immaterial.

Example 2.1. *For* $m = 1, 2, 3, \cdots, n = 1, 2, 3, \cdots$, *let*

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n,

$$\lim_{m \to \infty} s_{m,n} = 1,$$

so that

$$\lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} = 1.$$

On the other hand, for every fixed m*,*

$$\lim_{n \to \infty} s_{m,n} = 0,$$

so that

$$\lim_{m \to \infty} \lim_{n \to \infty} s_{m,n} = 0.$$

Example 2.2. *Let*

$$f_n(x) = \frac{x^2}{(1+x^2)^n}, \quad x \in \mathbb{R}; n = 0, 1, 2, \cdots,$$