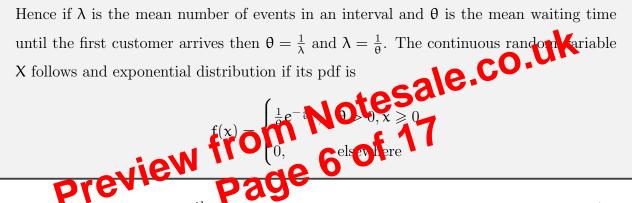
$$F(w) = p(W \le w)$$

=1-p(W > w)
=1-p(no customers arrive in [0,w])
=1-p(X = 0 with mean λw)
=1- $\frac{e^{-\lambda w}(\lambda w)^0}{0!}$
=1- $e^{-\lambda w}$ w > 0

NB: If the mean number of events in an interval of length 1 is λ then the mean number of events in the interval of length w is λw .

Therefore

$$f(w) = F'(w) = -e^{-\lambda w}(-\lambda) = \lambda e^{-\lambda w}, 0 < w < \infty$$



The waiting time until the α^{th} event follows a gamma distribution with parameters θ (mean waiting time until the first event) and α (number of events for which you are waiting to occur).

Illustration 2

Suppose X is the number of customers arriving at a bank in an interval of length 1.

X is a Poisson random variable. Let the mean number of events in the interval of length 1 be λ .

Let W denote the waiting time until the α^{th} customer arrives (i.e. until the α^{th} event occurs). It is a continuous random variable. What is the distribution of W? We can determine $F(w) = p(W \leq w)$? is a probability density function. The distribution of a random variable X with probability function as in Equation (24) is called the Standard Beta distribution with parameters α and β.

The Standard Beta distribution reduces to the Uniform (Rectangular) Distribution over the interval [0, 1] when $\alpha = 1$ and $\beta = 1$.

Definition: Beta Distribution

The case

The beta distribution provides positive density only for X in an interval of finite length.

A random variable X is said to have a beta distribution with parameters α , β (both positive), A and B if the pdf of X is

$$f(x; \alpha, \beta, A, B) = \begin{cases} \frac{1}{(B-A)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x-A}{B-A}\right)^{\alpha-1} \left(\frac{B-x}{B-A}\right)^{\beta-1}, & A \leq x \leq B\\ 0, & \text{elsewhere} \end{cases}$$
(25)
The case A = 0 and B = 1 gives the standard beta distribution.
The mean and variance of X are
$$\mu = E[x] = A + (B-A) \cdot \frac{\alpha}{\alpha+\beta}$$
(25)

The rth moment about the origin of the Standard Beta Distribution

The r^{th} moment (about the origin) of a probability distribution denoted by μ_r is given by

$$\mu_{\mathsf{r}} = \mathsf{E}(x^{\mathsf{r}}) = \int_{-\infty}^{\infty} x^{\mathsf{r}} \mathsf{f}(x) dx$$

For the Standard Beta distribution, the r^{th} moment (about the origin) is given by

$$\mu_{r} = \mathsf{E}[\mathbf{x}^{r}] = \int_{-\infty}^{\infty} \mathbf{x}^{r} \mathbf{f}(\mathbf{x}) d\mathbf{x}$$
$$= \int_{0}^{1} \mathbf{x}^{r} \frac{1}{\mathsf{B}(\alpha,\beta)} \mathbf{x}^{\alpha-1} (1-\mathbf{x})^{\beta-1} d\mathbf{x}$$
$$= \frac{1}{\mathsf{B}(\alpha,\beta)} \int_{0}^{1} \mathbf{x}^{\alpha+r-1} (1-\mathbf{x})^{\beta-1} d\mathbf{x}$$
$$= \frac{1}{\mathsf{B}(\alpha,\beta)} \mathsf{B}(\alpha+r,\beta)$$
$$= \frac{\mathsf{\Gamma}(\alpha+\beta)}{\mathsf{\Gamma}(\alpha)\mathsf{\Gamma}(\beta)} \frac{\mathsf{\Gamma}(\alpha+r)\mathsf{\Gamma}(\beta)}{\mathsf{\Gamma}(\alpha+r+\beta)}$$