$$
\begin{aligned}
F(w) =& p(W \leqslant w) \\
=& 1 - p(W > w) \\
=& 1 - p(\text{ no customers arrive in } [0, w]) \\
=& 1 - p(X = 0 \text{ with mean } \lambda w) \\
=& 1 - \frac{e^{-\lambda w}(\lambda w)^0}{0!} \\
=& 1 - e^{-\lambda w}, w > 0
\end{aligned}
$$

NB: If the mean number of events in an interval of length 1 is λ then the mean number of events in the interval of length w is λw.

Therefore

$$f(w) = F'(w) = -e^{-\lambda w}(-\lambda) = \lambda e^{-\lambda w}, 0 < w < \infty$$

Hence if λ is the mean number of events in an interval and θ is the mean waiting time until the first customer arrives then $\theta = \frac{1}{\lambda}$ and $\lambda = \frac{1}{\theta}$. The continuous random variable X follows and exponential distribution if its pdf is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{1}{\theta}x}, & \theta > 0, x \geqslant 0 \\ 0, & \text{elsewhere} \end{cases}$$

The waiting time until the α^{th} event follows a gamma distribution with parameters θ (mean waiting time until the first event) and α (number of events for which you are waiting to occur).

Illustration 2

Suppose X is the number of customers arriving at a bank in an interval of length 1.

X is a Poisson random variable. Let the mean number of events in the interval of length 1 be λ.

Let W denote the waiting time until the α^{th} customer arrives (i.e. until the α^{th} event occurs). It is a continuous random variable. What is the distribution of W?

We can determine $F(w) = p(W \leqslant w)$?

is a probability density function. The distribution of a random variable X with probability function as in Equation (24) is called the Standard Beta distribution with parameters α and β.

The Standard Beta distribution reduces to the Uniform (Rectangular) Distribution over the interval $[0, 1]$ when $\alpha = 1$ and $\beta = 1$.

Definition: Beta Distribution

The beta distribution provides positive density only for X in an interval of finite length.

A random variable X is said to have a beta distribution with parameters α, β (both positive), A and B if the pdf of X is

$$f(x;\alpha,\beta,A,B) = \begin{cases} \frac{1}{(B-A)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{x-A}{B-A}\right)^{\alpha-1}\left(\frac{B-x}{B-A}\right)^{\beta-1}, & A \leqslant x \leqslant B \\ 0, & \text{elsewhere} \end{cases} \tag{25}$$

The case $A = 0$ and $B = 1$ gives the standard beta distribution.

The mean and variance of X are

$$\mu = E[x] = A + (B-A).\frac{\alpha}{\alpha+\beta}$$

$$\sigma^2 = Var(X) = \frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

The r^{th} moment about the origin of the Standard Beta Distribution

The r^{th} moment (about the origin) of a probability distribution denoted by μ_r is given by

$$\mu_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

For the Standard Beta distribution, the r^{th} moment (about the origin) is given by

$$\begin{aligned}
\mu_r = E[x^r] &= \int_{-\infty}^{\infty} x^r f(x)dx \\
&= \int_0^1 x^r \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+r-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+r,\beta) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+r)\Gamma(\beta)}{\Gamma(\alpha+r+\beta)}
\end{aligned}$$