2. ONE-SIDED LIMITS

Sometimes the values of a function f(x) tend to different limits as x approaches a number c from different sides. When this happens, we call the limit of f(x) as x approaches c from the right, **the right-hand limit of f(x) at c.** Similarly we call the limit of f(x) as x approaches c from the left, **the left-hand limit of f(x) at c.** Limits such as these are referred to as **ONE-SIDED LIMITS.**

Definition "Informal":

Let L be a real number

(i) Suppose that f(x) is defined near c for x > c, and that as x gets close to c, f(x) gets close to L. Then we say L is the **right-hand limit of f(x)** as x approaches c from the right and we write

$$\lim_{x \to c^+} f(x) = L$$

(ii) Suppose that f(x) is defined near c for x < c and that as x gets close to c, f(x) gets close to L. Then we say that L is the **left-hand limit** of f(x) as x approaches c from the left, and we write

$$\lim_{x \to c^-} f(x) = L$$

Theorem 2.1
The two-sided limit

$$\lim_{x \to c} f(x) = L$$

if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

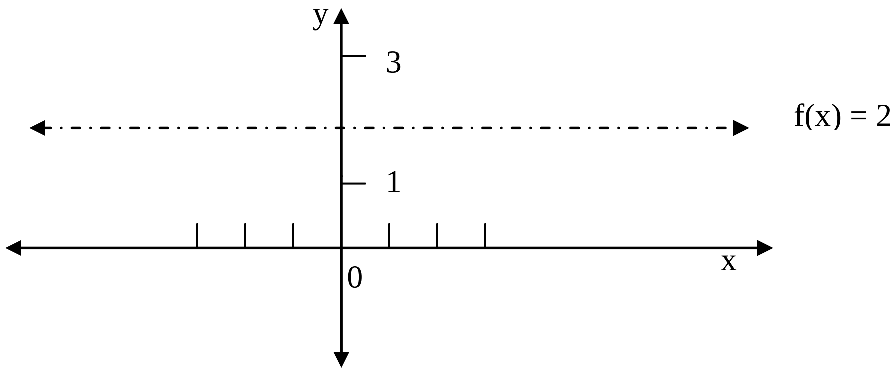
Examples

(a) Find $\lim_{t \to 0} g(t)$ if $g(t) = \begin{cases} 2t^2 + 4 & ; \quad t \le 0 \\ (t-2)^2 & ; \quad t > 0 \end{cases}$

Solution:

$$\lim_{t \to 0^-} g(t) = \lim_{t \to 0^-} (2t^2 + 4)$$

We can show that the function f(x) = 2 is continuous by drawing the curve of f(x).



We cannot always rely on the drawing of graphs for continuity of functions, because sometimes it is difficult, if not impossible, to draw some of the graphs. We need a general mathematical definition.

Definition 2.2:
A function f(x) is continuous at c if the following three conditions are satisfied:

(i) $f(c)$ is defined, where c lies in the domain of f.

(ii) $\lim_{x \to c} f(x)$ exists

(iii) $\lim_{x \to c} f(x) = f(c)$

Remark:
The limit in the definition above is to be a two-sided limit.

Examples:

1. Determine whether the function f defined as $f(x) = \begin{cases} x - 1 & ; \quad 0 \le x < 1 \\ 1 & ; \quad x = 1 \\ 2x - 2 & ; \quad x > 1 \end{cases}$

   is continuous at x = 1.

Solution:

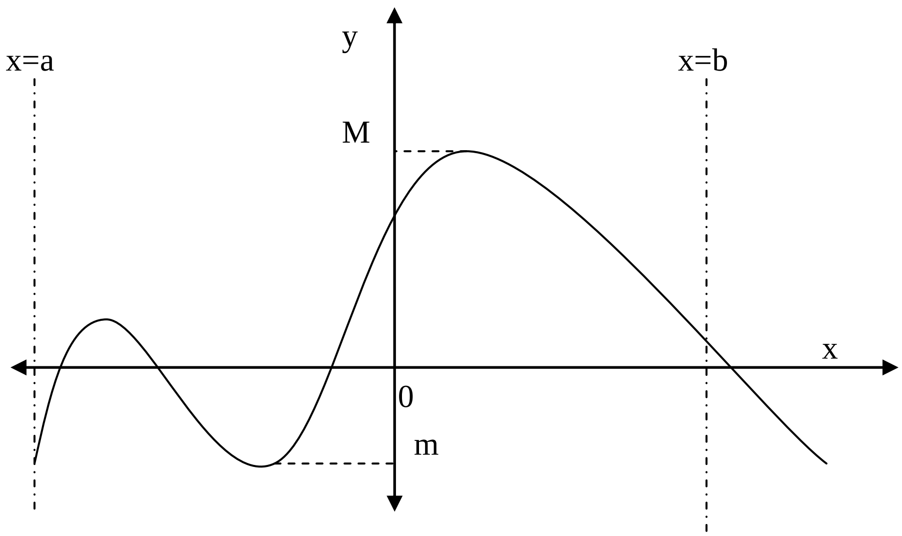
(i) $f(1) = 1$

(ii) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1)$
$= 1 - 1$
$= 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 2)$
$= 2 - 2 = 0$

Therefore $\lim_{x \to 1} f(x) = 0$ since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

(iii) Since $f(1) \neq \lim_{x \to 1} f(x)$, we conclude that the function f is not continuous at x = 1.



REMARKS:

1. The theorem above is in general not true on an open interval (a,b).

   For example, $f(x) = \frac{1}{x}$ is continuous on (0,1), but as x approaches 0 from the right, f(x) becomes very large and f is therefore not bounded on that interval. Similarly, f(x) = x has the least upper bound 1 in (0,1), but there is no $x \in (0,1)$, such that f(x) = 1.

2. The theorem above is generally also not true if f is not continuous in [a,b].

   For example, $f(x) = \frac{1}{x}$ is not bounded on the interval [-1,1] because it is not continuous at x = 0.

Theorem 2.8 (Intermediate Value Theorem)
Let f be a continuous function on [a,b]. If k is any number between f(a) and f(b), there exists a number $c \in (a,b)$ such that f(c) = k.