If $\mu(F_1, \ldots, F_k, \widehat{F}_{k+1}) > 0$, then $F_1^0(x), \ldots, F_k^0(x), \widehat{F}_{k+1}^0$ are also functionally dependent. Continuing the above the procedure, finally we can get a polynomial $\widetilde{P}(z)$ with $z = (z_1, \ldots, z_{k+1})$ such that

$$F_1(x), \ldots, F_k(x), \widetilde{F}_{k+1}(x) = \widetilde{P}(F_1(x), \ldots, F_{k+1}(x)),$$

are functionally independent and $\mu(F_1, \ldots, F_k, \widetilde{F}_{k+1}) = 0$. The last equality implies that the rational functions $F_1^0, \ldots, F_k^0, \widetilde{F}_{k+1}^0$ are functionally independent. Furthermore, the generalized rational functions

$$F_1(x), \ldots, F_k(x), \widetilde{F}_{k+1}(x), F_{k+2}(x), \ldots, F_m(x),$$

are functionally independent, because $\widetilde{F}_{k+1}(x)$ involves only $F_1, \ldots, F_{k+1}$, and $F_1, \ldots, F_k, \widetilde{F}_{k+1}$ are functionally independent, and also $F_1, \ldots, F_m$ are functionally independent. This can also be obtained by direct calculations as follows

$$\begin{aligned}
&\det\left(\frac{\partial(F_1(x), \ldots, F_k(x), \widetilde{F}_{k+1}(x), F_{k+2}(x), \ldots, F_m(x))}{\partial(x_1, \ldots, x_n)}\right)\\
= \quad &\det\begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\
\vdots & \vdots & \vdots\\
\frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n}\\
\frac{\partial \widetilde{P}}{\partial z_1}\frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial \widetilde{P}}{\partial z_{k+1}}\frac{\partial F_{k+1}}{\partial x_1} & \cdots & \frac{\partial \widetilde{P}}{\partial z_1}\frac{\partial F_1}{\partial x_n} + \ldots + \frac{\partial \widetilde{P}}{\partial z_{k+1}}\frac{\partial F_{k+1}}{\partial x_n}\\
\frac{\partial F_{k+2}}{\partial x_1} & \cdots & \frac{\partial F_{k+2}}{\partial x_n}\\
\vdots & \vdots & \vdots\\
\frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n}
\end{pmatrix}\\
= \quad &\frac{\partial \widetilde{P}}{\partial z_{k+1}}(x)\det\left(\frac{\partial(F_1(x), \ldots, F_m(x))}{\partial(x_1, \ldots, x_n)}\right) \neq 0.
\end{aligned}$$

If $k+1 = m$, the proof is completed. Otherwise we can continue the above procedure, and finally we get the functionally independent generalized rational functions $F_1(x), \ldots, F_k(x), \widetilde{F}_{k+1}(x) = \widetilde{P}_{k+1}(F_1(x), \ldots, F_{k+1}(x)), \ldots, \widetilde{F}_m(x) = \widetilde{P}_m(F_1(x), \ldots, F_m(x))$ such that their lowest order rational functions $F_1^0(x), F_k^0(x), \widetilde{F}_{k+1}^0(x), \ldots, \widetilde{F}_m^0(x)$ are functionally independent, where $\widetilde{P}_j$ for $j = k+1, \ldots, m$ are polynomials in $F_1, \ldots, F_j$.

The proof of the lemma is completed. $\square$

The next result characterizes rational first integrals of system (1). A *rational monomial* is by definition the ratio of two monomials, i.e. of the form $x^{\mathbf{k}}/x^{\mathbf{l}}$ with $\mathbf{k}, \mathbf{l} \in (\mathbb{Z}^+)^n$. The rational monomial $x^{\mathbf{k}}/x^{\mathbf{l}}$ is *resonant* if $\langle \lambda, \mathbf{k} - \mathbf{l} \rangle = 0$. A rational function is *homogeneous* if its denominator and numerator are both homogeneous polynomials. A rational homogeneous function is *resonant* if the ratio of any two elements in the set of all

So there exists a constant c such that

$$\left\langle \partial_x G^0(x), Ax \right\rangle - cG^0(x) \equiv 0, \quad \left\langle \partial_x H^0(x), Ax \right\rangle - cH^0(x) \equiv 0.$$

Set $\deg G^0(x) = l$, $\deg H^0(x) = m$ and L_c be the linear operator defined in Lemma 8. Recall from Lemma 8 that L_c has respectively the spectrums on $\mathcal{H}_n^l$

$$\mathcal{S}_l := \{\langle \mathbf{l}, \lambda\rangle - c : \ \mathbf{l} \in (\mathbb{Z}^+)^n, \ |\mathbf{l}| = l\},$$

and on $\mathcal{H}_n^m$

$$\mathcal{S}_m := \{\langle \mathbf{m}, \lambda\rangle - c : \ \mathbf{m} \in (\mathbb{Z}^+)^n, \ |\mathbf{m}| = m\}.$$

Separate $\mathcal{H}_n^l = \mathcal{H}_{n1}^l + \mathcal{H}_{n2}^l$ in such a way that for any $p(x) \in \mathcal{H}_{n1}^l$ its monomial $x^{\mathbf{l}}$ satisfies $\langle \mathbf{l}, \lambda\rangle - c = 0$, and for any $q(x) \in \mathcal{H}_{n2}^l$ its monomial $x^{\mathbf{l}}$ satisfies $\langle \mathbf{l}, \lambda\rangle - c \neq 0$. Separate $G^0(x)$ in two parts $G^0(x) = G_1^0(x) + G_2^0(x)$ with $G_1^0 \in \mathcal{H}_{n1}^l$ and $G_2^0 \in \mathcal{H}_{n2}^l$. Since A is in its Jordan normal form and is lower triangular, it follows that

$$L_c \mathcal{H}_{n1}^l \subset \mathcal{H}_{n1}^l, \quad \text{and} \quad L_c \mathcal{H}_{n2}^l \subset \mathcal{H}_{n2}^l.$$

Hence $L_c G^0(x) \equiv 0$ is equivalent to

$$L_c G_1^0(x) \equiv 0 \quad \text{and} \quad L_c G_2^0(x) \equiv 0.$$

Since L_c has the spectrum without zero element on $\mathcal{H}_{n2}^l$ and so it is invertible on $\mathcal{H}_{n2}^l$, the equation $L_c G_2^0(x) \equiv 0$ has only the trivial solution, i.e. $G_2^0(x) \equiv 0$. This proves that $G^0(x) = G_1^0(x)$, i.e. each monomial, say $x^{\mathbf{l}}$, of $G^0(x)$ satisfies $\langle \mathbf{l}, \lambda\rangle - c = 0$.

Similarly we can prove that each monomial, say $x^{\mathbf{m}}$, of $H^0(x)$ satisfies $\langle \mathbf{m}, \lambda\rangle - c = 0$. This implies that $\langle \mathbf{l} - \mathbf{m}, \lambda\rangle = 0$. The above proofs show that $F^0(x) = G^0(x)/H^0(x)$ is a resonant rational homogeneous first integral of $\mathcal{X}_1$. □

Having the above lemmas we can prove Theorem 1.

Proof of Theorem 1. Let

$$F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)},$$

be the m functionally independent generalized rational first integrals of $\mathcal{X}$. Since the polynomial functions of $F_i(x)$ for $i = 1, \ldots, m$ are also generalized rational first integrals of $\mathcal{X}$, so by Lemma 6 we can assume without loss of generality that

$$F_1^0(x) = \frac{G_1^0(x)}{H_1^0(x)}, \ldots, F_m^0(x) = \frac{G_m^0(x)}{H_m^0(x)},$$

are functionally independent.

Lemma 7 shows that $F_1^0(x), \ldots, F_m^0(x)$ are resonant rational homogeneous first integrals of the linear vector field $\mathcal{X}_1$, that is, these first integrals are rational functions in the variables given by resonant rational monomials. According to the linear algebra (see for instance [3]), the square matrix A

Equations (23) are equivalent to

$$\begin{aligned}&\widetilde{H}_i^0(t,y)\left(\partial_t\widetilde{G}_i^0(t,y)+\left\langle\partial_y\widetilde{G}_i^0(t,y),\Lambda y\right\rangle\right)\\ &\equiv\widetilde{G}_i^0(t,y)\left(\partial_t\widetilde{H}_i^0(t,y)+\left\langle\partial_y\widetilde{H}_i^0(t,y),\Lambda y\right\rangle\right),\qquad i=1,\ldots,m.\end{aligned}\tag{25}$$

So there exist constants, say c_i, such that

$$\partial_t\widetilde{G}_i^0(t,y)+\left\langle\partial_y\widetilde{G}_i^0(t,y),\Lambda y\right\rangle-c_i\widetilde{G}_i^0(t,y)\equiv 0,\tag{26}$$

and

$$\partial_t\widetilde{H}_i^0(t,y)+\left\langle\partial_y\widetilde{H}_i^0(t,y),\Lambda y\right\rangle-c_i\widetilde{H}_i^0(t,y)\equiv 0.\tag{27}$$

For the set of monomials of degree k, $\Upsilon_k:=\left\{y^{\mathbf{k}}:\mathbf{k}\in(\mathbb{Z}^+)^n,\,|\mathbf{k}|=k\right\}$, we define their order as follows: $y^{\mathbf{p}}$ is before $y^{\mathbf{q}}$ if $\mathbf{p}-\mathbf{q}\succ 0$, i.e., there exists an $i_0\in\{1,\ldots,n\}$ such that $p_i=q_i$ for $i=1,\ldots,i_0-1$ and $p_{i_0}>q_{i_0}$. Then Υ_k is a base of the set of homogeneous polynomials of degree k with the given order. According to the given base and order, each homogeneous polynomial of degree k is uniquely determined by its coefficients.

We denote by $\widetilde{G}_i^0(t)$ the vector of dimension $\begin{pmatrix} l_i+n-1\\ n-1\end{pmatrix}$ formed by the coefficients of $\widetilde{G}_i^0(t,y)$. Let $\mathcal{L}_k$ be the linear operator on $\mathcal{H}_n^k(t)$, the linear space of homogeneous polynomials of degree k in y with coefficients 2π periodic in t, defined by

$$\mathcal{L}_k(h(t,y))=\langle\partial_y h(t,y),\Lambda y\rangle,\quad h(t,y)\in\mathcal{H}_n^k(t).$$

Using these notations equations (26) and (27) can be written as

$$\partial_t\widetilde{G}_i^0(t)+(\mathcal{L}_{l_i}-c_i)\widetilde{G}_i^0(t)\equiv 0,\quad \partial_t\widetilde{H}_i^0(t)+(\mathcal{L}_{m_i}-c_i)\widetilde{H}_i^0(t)\equiv 0.$$

They have solutions

$$\widetilde{G}_i^0(t)=\exp\left((c_i\mathbf{E}_{1i}-\mathcal{L}_{l_i})t\right)\widetilde{G}_i^0(0),\quad \widetilde{H}_i^0(t)=\exp\left((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})t\right)\widetilde{H}_i^0(0),$$

where $\mathbf{E}_{1i}$ and $\mathbf{E}_{2i}$ are two identity matrices of suitable orders. In order that $\widetilde{G}_i^0(t)$ and $\widetilde{H}_i^0(t)$ be 2π periodic, we should have

$$\left(\exp\left((c_i\mathbf{E}_{1i}-\mathcal{L}_{l_i})2\pi\right)-\mathbf{E}_{1i}\right)\widetilde{G}_i^0(0)=0,\tag{28}$$

and

$$\left(\exp\left((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})2\pi\right)-\mathbf{E}_{2i}\right)\widetilde{H}_i^0(0)=0.\tag{29}$$

Recall that $\lambda=(\lambda_1,\ldots,\lambda_n)$ be the eigenvalues of Λ. Then it follows from Lemma 8 that $\exp((c_i\mathbf{E}_{2i}-\mathcal{L}_{l_i})2\pi)$ and $\exp((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})2\pi)$ have respectively the eigenvalues

$$\left\{\exp((c_i-\langle\mathbf{l},\lambda\rangle)2\pi):\,\mathbf{l}\in(\mathbb{Z}^+)^n,\,|\mathbf{l}|=l_i\right\},$$

and

$$\left\{\exp((c_i-\langle\mathbf{m},\lambda\rangle)2\pi):\,\mathbf{m}\in(\mathbb{Z}^+)^n,\,|\mathbf{m}|=m_i\right\}.$$