$\langle \phi_{\mathbf{x}_0}, \mathsf{H}_e(z)^{-1}\phi_{\mathbf{y}_0}\rangle$ can approach the Green's function:

$$\begin{aligned}\langle \phi_{\mathbf{x}_0}, \mathsf{H}_e(z)^{-1}\phi_{\mathbf{y}_0}\rangle \\ &= \int_{\mathbb{R}^3} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{y}\, \phi_a(\mathbf{x}-\mathbf{x}_0)\mathsf{G}_e(\mathbf{x},\mathbf{y};z)\phi_a(\mathbf{y}-\mathbf{y}_0) \\ &\approx \mathsf{G}_e(\mathbf{x}_0,\mathbf{y}_0;z)\,.\end{aligned} \tag{39}$$

Here, it is stressed that nothing authorizes to take the limit $a \downarrow 0$ since the function ϕ_a is not square integrable at this limit. Thus the coefficient $\langle \phi_{\mathbf{x}_0}, \mathsf{H}_e(z)^{-1}\phi_{\mathbf{y}_0}\rangle$ just allows to address an approximation of the Green's function $\mathsf{G}_e(\mathbf{x}_0,\mathbf{y}_0;z)$.

According to these arguments above, it is assumed that the electromagnetic Green's function can be defined, and that its properties can be established from the coefficients $\langle \phi, \mathsf{H}_e(z)^{-1}\psi\rangle$. First, it is clear that all the analytic properties of $\mathsf{H}_e(z)^{-1}$ and $\mathsf{H}(z,\xi)^{-1}$ are directly transposable to coefficients $\langle \phi, \mathsf{H}_e(z)^{-1}\psi\rangle$ and $\langle \phi, \mathsf{H}(z,\xi)^{-1}\psi\rangle$. Indeed, it is enough to expand the inverse in $\langle \phi, \mathsf{H}_e(z)^{-1}\psi\rangle$ in a power series like (29) and then to check that the resulting power series with the coefficients converges. Another important properties of the Green's function is the behavior for large frequency z. First, it is shown in the appendix A that it decreases like $1/(z^2\varepsilon_0\mu_0)$ or, equivalently,

$$\lim_{|z|\to\infty} z^2\varepsilon_0\mu_0 \langle \phi, \mathsf{H}_e(z)^{-1}\psi\rangle = \langle \phi, \psi\rangle\,. \tag{40}$$

It has to be noticed that this asymptotic behavior is for the modulus $|z|$ of the complex frequency which tends to infinity. An important asymptotic regime is the limit $\omega \to \infty$ in the complex frequency z, i.e. for fixed imaginary part $\mathrm{Im}(z)$. In this case one can show that

$$z^2\left[\mathsf{H}_e(z)^{-1} - \mathsf{H}_0(z)^{-1}\right] \tag{41}$$

is bounded when $\omega \to \infty$. This can be established writting the difference

$$\mathsf{H}_e(z)^{-1} - \mathsf{H}_0(z)^{-1} = -\mathsf{H}_0(z)^{-1} z^2\mu_0[\varepsilon(\mathbf{x},z)-\varepsilon_0]\mathsf{H}_e(z)^{-1}\,, \tag{42}$$

and then using the bound (32) and the estimate (18). Hence it is found that

$$\begin{aligned} z^2\left[\mathsf{H}_e(z)^{-1} - \mathsf{H}_0(z)^{-1}\right] \\ &\underset{\omega\to\infty}{\approx} z\mathsf{H}_0(z)^{-1}\,[\partial_t\chi](\mathbf{x},0^+)\,z\mathsf{H}_e(z)^{-1} \\ &\leq \frac{[\partial_t\chi](\mathbf{x},0^+)}{[\varepsilon_0\mu_0\,\mathrm{Im}(z)]^2}\,. \end{aligned} \tag{43}$$

These properties and asymptotic behaviors will be used in the next section to derive a version of Kramers-Kronig relations for the electromagnetic Green's function.

IV. KRAMERS-KRONIG RELATIONS FOR THE ELECTROMAGNETIC GREEN'S FUNCTION

The Kramers-Kronig relations can be applied to all function derived from a causal signal. It is generally used to analyze the dielectric permittivity, the permeability or the optical index[1,3]. A new version of Kramers-Kronig relations, given by equation (10), has been proposed recently[7,10]. This version shows that the general expression of the permittivity is a continuous superposition of elementary resonances given by the elastically bound electron model. Thus it extends the classical Drude-Lorentz expression[1] of the permittivity, and also its quantum mechanical justification based on the electric dipole approximation[11]. In particular, the continuous superposition of resonances in (10) describes a regime with absorption, while the quantum mechanics model is reduced to a discrete superposition of resonances and thus to the description of systems without absorption.

In this section, it is proposed to express the new version of Kramers-Kronig relations for the electromagnetic Green's function or, equivalently, for the inverse operator $\mathsf{H}_e(z)^{-1}$. The objective is to transpose all the properties of the permittivity, and to make it possible to use all the knowledge on permittivity $\varepsilon(\mathbf{x},z)$ for the electromagnetic Green's function.

The inverse operators $\mathsf{H}_e(z)^{-1}$ and $\mathsf{H}_0(z)^{-1}$ are expected to have properties similar to those of permittivities $\varepsilon(\mathbf{x},z)$ and ε_0. Thus the following operator is considered:

$$\mathsf{R}(z) = \mathsf{H}_e(z)^{-1} - \mathsf{H}_0(z)^{-1}\,, \tag{44}$$

First, it is noticed that, as well as $\mathsf{H}_e(z)^{-1}$ and $\mathsf{H}_0(z)^{-1}$, the adjoint operator of $\mathsf{R}(z)^{-1}$ is

$$\left[\mathsf{R}(z)^{-1}\right]^\dagger = \mathsf{R}(-\overline{z})^{-1}\,, \tag{45}$$

which is related to $\overline{\varepsilon(z)} = \varepsilon(-\overline{z})$. Next, let the operator $\mathsf{X}(t)$ be defined by

$$\mathsf{X}(t) = \int_{\Gamma_\eta} dz\, \exp[-izt]\,\mathsf{R}(z)\,, \tag{46}$$

where Γ_η is the horizontal line parallel to the real axis at a distance η, of complex numbers $z = \omega + i\eta$ with $\eta > 0$. It is stressed that this integral is well defined since, thanks to (43), $\mathsf{R}(z)$ is bounded and decreases like $1/\omega^2$. Also, this decrease in $1/\omega^2$ implies that $\mathsf{X}(t)$ is the Fourier transform of an integrable function, and thus $\mathsf{X}(t)$ is continuous of t. The integral expression of $\mathsf{X}(t)$ is independent of η thanks to the analytic nature of the operator under the integral. The operator $\mathsf{X}(t)$ is selfadjoint since, for $z = \omega + i\eta$,

$$\mathsf{X}(t)^\dagger = \int_{\mathbb{R}} d\omega\, \exp[i\overline{z}t]\,\mathsf{R}(-\overline{z}) = \int_{\mathbb{R}} d\omega\, \exp[-izt]\,\mathsf{R}(z)\,, \tag{47}$$

where (45) has been used and the change $\omega \to -\omega$ has been performed to obtain the last expression. In addition, it can be checked that $\mathsf{X}(t)$ vanishes for negative times. Indeed, in that case, the integral (46) can be computed by closing the line Γ_η by a semi circle with infinite radius in the upper half plane. Since all the functions are