This theorem neither requires positivity nor Lorentz covariance. It expresses a property of the domain of holomorphy of the Wightman functions, and of the boundary values from this domain. In fact, it states that appropriate boundary values of the $(m+n)-$point holomorphic function $\mathfrak{W}_{m+n}$, taken in the region where all the variables $w_1, \dots,\ w_m,\ z_1, \dots,\ z_n$ belong to $W_{(r)} \cap X_d$, are holomorphic with respect to the group variable $\lambda$ (for $\lambda \in \mathbf{C} \setminus \mathbf{R}_+$) in the orbits $(w, x) \mapsto (w, [\lambda] x)$ of $T^{(c)}_{h(x_0)}$ (with $w = (w_1, \dots,\ w_m),\ \ x = (x_1, \dots,\ x_n), \lambda = e^{\frac{t}{R}}$) and such that:

for $\lambda > 0$,

$$\mathfrak{W}_{m+n}(w,\ [\lambda + i0]x) = \mathrm{W}_{m+n}(w,\ [\lambda]x), \quad \mathfrak{W}_{m+n}(w,\ [\lambda - i0]x) = \mathrm{W}_{m+n}([\lambda]x,\ w) \tag{47}$$

and for $\lambda < 0$, putting $x_{\leftarrow} = (x_n, \dots,\ x_1)$,

$$\mathfrak{W}_{m+n}(w,\ [\lambda]x) = \mathrm{W}_{m+n}(w,\ [\lambda]x_{\leftarrow}) = \mathrm{W}_{m+n}([\lambda]x_{\leftarrow},\ w) \tag{48}$$

the latter equality being a direct consequence of locality (since $x \in W^n_{(r)}$ and $\lambda < 0$ imply $[\lambda]x_{\leftarrow} \in W^n_{(l)}$).

The theorem will be proved here under the simplifying assumption that the temperedness condition (17) holds.

**Proof**

Four permuted branches of the function $\mathfrak{W}_{m+n}$ are involved in the proof. The variables $w = (w_1, \dots,\ w_m)$ will always be kept real in $W^m_{(r)} \cap X^m_d$, while the variables $z = (z_1, \dots,\ z_n)$ are complex (in $X^{(c)n}_d$) and we denote $y = \mathrm{Im}\, z$. The corresponding analyticity domains in the variables $z$ (described below) are obtained in the boundaries (i.e. in the "face" $\mathrm{Im}\, w = 0$) of four permuted tuboids $\mathcal{T}^\pi_{m+n}$ according to the prescription of our weak spectral condition. In view of the distribution boundary value procedure, restricted to the subset of variables $w$, these analyticity domains are obtained whenever one smears out the permuted functions $\mathrm{W}^\pi_{m+n}$ under consideration with a fixed function $f_m \in \mathcal{D}(W^m_{(r)} \cap X^m_d)$. (this function being understood as *the* function named $f_m$ in the statement of the theorem). These four branches are:



i) $\mathrm{W}_{m+n}(w_1, \dots,\ w_m,\ z_1, \dots,\ z_n) = \mathrm{W}_{m+n}(w,\ z)$, holomorphic in the tuboid:
$\mathcal{Z}_{n\,+} = \left\{ z \in X^{(c)n}_d;\ y_1 \in V_+, y_j - y_{j-1} \in V_+,\ j = 2, \dots,\ n \right\}$;

ii) $\mathrm{W}_{m+n}(z_n, \dots,\ z_1,\ w_1, \dots,\ w_m) = \mathrm{W}_{m+n}(z_{\leftarrow},\ w)$, holomorphic in the opposite tuboid:
$\mathcal{Z}_{n\,-} = \{ z \in X^{(c)n}_d;\ y_1 \in V_-, y_j - y_{j-1} \in V_-,\ j = 2, \dots,\ n \}$;

iii) $\mathrm{W}_{m+n}(z_1, \dots,\ z_n,\ w_1, \dots,\ w_m) = \mathrm{W}_{m+n}(z,\ w)$, holomorphic in the tuboid:
$\mathcal{Z}'_{n\,+} = \{ z \in X^{(c)n}_d;\ y_n \in V_-, y_j - y_{j-1} \in V_+, j = 2, \dots, n \}$;

iv) $\mathrm{W}_{m+n}(w_1, \dots,\ w_m,\ z_n, \dots,\ z_1) = \mathrm{W}_{m+n}(w,\ z_{\leftarrow})$, holomorphic in the opposite tuboid:
$\mathcal{Z}'_{n\,-} = \{ z \in X^{(c)n}_d;\ y_n \in V_+, y_j - y_{j-1} \in V_-, j = 2 \dots,\ n \}$.

Correspondingly, with the fixed function $f_m \in \mathcal{D}(W^m_{(r)} \cap X^m_d)$ we associate the following four functions $z \mapsto F_{\pm}(f_m;\ z)$ and $z \mapsto F'_{\pm}(f_m;\ z)$:

$$F_+(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(w,\ z)\ f_m(w)\ d^m\sigma(w), \qquad F_-(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(z_{\leftarrow},\ w)\ f_m(w)\ d^m\sigma(w) \tag{49}$$

$$F'_+(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(z,\ w)\ f_m(w)\ d^m\sigma(w), \qquad F'_-(f_m;\ z) = \int_{X^m_d} \mathrm{W}_{m+n}(w,\ z_{\leftarrow})\ f_m(w)\ d^m\sigma(w) \tag{50}$$

which are respectively holomorphic in $\mathcal{Z}_{n\,+},\ \mathcal{Z}_{n\,-},\ \mathcal{Z}'_{n\,+}$ and $\mathcal{Z}'_{n\,-}$. By letting the variables $z$ tend to the reals from the respective tuboids $\mathcal{Z}_{n\,+},\ \mathcal{Z}_{n\,-},\ \mathcal{Z}'_{n\,+}$ and $\mathcal{Z}'_{n\,-}$, and taking the corresponding

Since the function $F(f_m;\ z)$ satisfies the analyticity and temperedness properties of the function $H(z)$ of Lemma 2 a), it follows that one can take the boundary value onto $X_d^n \times \mathbf{C}_+$ from $\mathcal{Z}_{n+} \times \mathbf{C}_+$ of the holomorphic function $(z,\ \lambda) \mapsto F(f_m;\ [\lambda]z)$ and obtain for every $g_n \in \mathcal{D}(X_d^n)$ the following relations (deduced from Eq. (55) after taking into account Eqs. (51) and (52)):

$$\int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ f_m \otimes g_{n\lambda} \rangle \quad \text{for } \lambda > 0, \tag{57}$$

$$\int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ g^{\leftarrow}_{n\ \lambda} \otimes f_m \rangle \quad \text{for } \lambda < 0. \tag{58}$$

Similarly, one can apply the results of Lemma 2 b) to the function $H'(z) = F'(f_m;\ z)$; one can thus take the boundary value onto $X_d^n \times \mathbf{C}_-$ from $\mathcal{Z}'_{n+} \times \mathbf{C}_-$ of the holomorphic function $(z,\ \lambda) \mapsto F'(f_m;\ [\lambda]z)$ and obtain for every $g_n \in \mathcal{D}(X_d^n)$ the following relations (deduced from Eq. (56) after taking into account Eqs. (53) and (54)):

$$\int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ g_{n\lambda} \otimes f_m \rangle \quad \text{for } \lambda > 0, \tag{59}$$

$$\int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) = \langle \mathcal{W}_{m+n},\ f_m \otimes g^{\leftarrow}_{n\ \lambda} \rangle \quad \text{for } \lambda < 0. \tag{60}$$

The l.h.s. of Eqs. (57) (or (58)) and (59) (or (60)) are respectively the boundary values of the holomorphic functions

$$G_{(f_m,g_n)}(\lambda) = \int_{X_d^n} F(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) \tag{61}$$

defined for $\lambda \in \mathbf{C}_+$ and

$$G'_{(f_m,g_n)}(\lambda) = \int_{X_d^n} F'(f_m;\ [\lambda]x) g_n(x) d^n\sigma(x) \tag{62}$$

defined for $\lambda \in \mathbf{C}_-$. For an arbitrary function $g_n \in \mathcal{D}(X_d^n)$, these two holomorphic functions are distinct from each other. Now, if $g_n$ is taken in $\mathcal{D}(\mathcal{U}^n_{h(x_0)})$, the r.h.s. of Eqs. (58) and (60) coincide in view of local commutativity, and therefore these two holomorphic functions admit a common holomorphic extension $G_{(f_m,g_n)}(\lambda)$ in $\mathbf{C} \setminus \mathbf{R}_+$ whose boundary values on $\mathbf{R} \setminus 0$ satisfy the properties a) and b) of the theorem. (in view of Eqs. (57)—(60)).

**Proof of Lemma 2**

We concentrate on part a) of the lemma, part b) being completely similar. At first, the fact that the function $(z,\ \lambda) \mapsto H([\lambda]z)$ can be analytically continued in $\mathcal{Z}_{n+} \times \mathbf{C}_+$ is a result of purely geometrical nature (based on the tube theorem) which can be obtained as a direct application of lemma 3 (ii) of Appendix A. In fact, for each point $x \in \mathcal{J}_n^{(r)}$, the set $\{z = [\lambda]x;\ \lambda \in \mathbf{C}_+\}$ is contained in $\Delta$ (namely in $\mathcal{Z}_{n+}$, as it directly follows from Eq. (3) and from the definitions of $\mathcal{J}_n^{(r)}$ and $\mathcal{Z}_{n+}$). One can even check that each point $x \in \mathcal{J}_n^{(r)}$ is on the edge of a small open tuboid $\tau(x)$ contained in $\mathcal{Z}_{n+}$ such that $\{z = [\lambda]z';\ z' \in \tau(x),\ \lambda \in \mathbf{C}_+\} \subset \mathcal{Z}_{n+} \cup \mathcal{V} \subset \Delta$. On the other hand, for each point $z \in \mathcal{Z}_{n+}$ there exists a neighbourhood $\delta_+(z)$ of the real positive axis and a neighbourhood $\delta_-(z)$ of the real negative axis in the complex $\lambda$-plane, such that the set $\{[\lambda]z;\ \ \lambda \in \delta^+(z) \cup \delta^-(z)\}$ is contained in $\Delta$: for $\lambda \in \delta_+(z)$ and $\lambda \in \delta_-(z)$ the corresponding subsets are respectively contained in $\mathcal{Z}_{n+}$ and in $\mathcal{Z}_{n-}$. Therefore, the assumptions of lemma 3 (ii) of Appendix A are fulfilled (by choosing the set $Q$ of the latter as a subset of $\tau(x)$ and $D' = \mathcal{Z}_{n+}$ after an appropriate adaptation of the variables). In order to see that the new domain thus obtained (i.e. $\{z = [\lambda]z';\ z' \in \mathcal{Z}_{n+},\ \lambda \in \mathbf{C}_+\}$ yields an enlargement of $\Delta$, it is sufficient to notice that every real point $x$ such that at least one component $x_j - x_{j-1}$ is time-like is transformed by any complex transformation $[\lambda]$ into a point outside $\mathcal{Z}_{n\pm}$ and this is of course also true for all points $z \in \mathcal{Z}_{n+}$ tending to such real (boundary) points (the neighbourhoods $\delta^{\pm}(z)$ becoming arbitrarily thin

By composing $\sigma_n$ with $z_j = \mathrm{th}\,(\zeta_j/2)$, we obtain a self-conjugate holomorphic diffeomorphism $\tau_n$ of the tube

$$\{\zeta = \xi + i\eta \in \mathbf{C}^{nd} \ : \ |\eta_j| < \pi/2,\ 1 \le j \le nd\} \tag{105}$$

onto a complex neighborhood of $a_n$ in $X_d^{(c)\,n}$ such that $\tau_n(0) = a_n$ and the image of the tube

$$\Theta_n = \{\zeta = \xi + i\eta \in \mathbf{C}^{nd} \ : \ 0 < \eta_j < \pi/2,\ 1 \le j \le nd\}. \tag{106}$$

is contained in $\mathcal{Z}_n$. Let

$$B_{n\,m}(\zeta,\ \zeta') = A_{n\,m}(\tau_n(\zeta),\ \tau_m(\zeta')). \tag{107}$$

The functions $B_{n\,m}$ are holomorphic in $\Theta_n \times \Theta_m^*$. Since for $\zeta = \xi + i\eta \in \mathbf{C}$,

$$\mathrm{th}\,(\zeta/2) = \frac{\mathrm{sh}\,\xi + i\sin\eta}{2|\mathrm{ch}\,(\zeta/2)|^2}, \tag{108}$$

the $B_{n\,m}$ satisfy

$$|B_{n\,m}(\zeta,\ \zeta')| \le K' \sum_j \left(\frac{\mathrm{e}^{|\xi_j|}}{|\sin\eta_j|}\right)^r + K' \sum_j \left(\frac{\mathrm{e}^{|\xi'_j|}}{|\sin\eta'_j|}\right)^r, \quad \forall \zeta = \xi + i\eta \in \Theta_n,\ \zeta' = \xi' + i\eta' \in \Theta_m^*. \tag{109}$$

They have boundary values $B_{n\,m}^{(v)}$ in the sense of generalized functions over test-functions of faster than exponential decrease. These boundary values satisfy, for each finite sequence $\{f_n\}$, $f_0 \in \mathbf{C}$, $f_n \in \mathcal{D}(\mathbf{R}^{nd})$ for $n \ge 1$,

$$\sum_{n,\ m} \int B_{n\,m}^{(v)}(\xi,\ \xi')\, f_n(\xi)\, \overline{f_m(\xi')}\, d\xi\, d\xi' \ge 0. \tag{110}$$

Let now

$$\rho_{n,\ \varepsilon}(\xi) = C(\varepsilon)\ \exp\left(-\sum_{j=1}^{nd} (\xi_j^2/\varepsilon)\right), \tag{111}$$

where $C(\varepsilon)$ is chosen so that $\int \rho_{n,\ \varepsilon}(\xi)\, d\xi = 1$. For each $\mu \in \mathbf{C}^{nd}$, the function $\xi \mapsto \rho_{n,\ \varepsilon}(\xi + \mu)$ is of gaussian decrease, and depends holomorphically on $\mu$. In particular if $\mu_n \in \Theta_n$, $\mu'_m \in \Theta_m^*$,

$$\begin{aligned} &\int_{\mathbf{R}^{nd}\times\mathbf{R}^{md}} B_{n\,m}^{(v)}(t,\ t')\, f_n(\xi)\, \rho_{n,\ \varepsilon}(t - \mu_n - \xi)\, \overline{f_m(\xi')\rho_{m,\ \varepsilon}(t' - \bar{\mu}'_m - \xi')}\, dt\, dt'\, d\xi\, d\xi' \\ &= \int_{\mathbf{R}^{nd}\times\mathbf{R}^{md}} B_{n\,m}(t + \mu_n,\ t' + \mu'_m)\, f_n(\xi)\, \rho_{n,\ \varepsilon}(t - \xi)\, \overline{f_m(\xi')\rho_{m,\ \varepsilon}(t' - \xi')}\, dt\, dt'\, d\xi\, d\xi', \end{aligned} \tag{112}$$

since both sides define analytic functions in $\Theta_n \times \Theta_m^*$ whose boundary values for real $\mu_n$, $\mu'_m$ coincide. The lhs satisfies the positivity conditions, by virtue of Eq. (110), if we chose $\mu'_n = \bar{\mu}_n$ for all $n$. It follows, by letting $\varepsilon$ tend to 0 in the rhs, that the functions $B_{n\,m}$ have the property (G.0) of Glaser's theorem 1 and therefore all the properties (G.0)-(G.4) in the sequence of domains $\{\Theta_n\}$. Coming back to the original variables, Glaser's theorem 1 now shows that the same properties, in particular (G.2), extend to the entire tuboid $\{\mathcal{Z}_n\}$. We have thus proved the following

**Proposition 5** *For any integer $M \ge 1$, there exist a sequence $\{F_{\nu,\ 0} \in \mathbf{C}\}_{\nu\in\mathbf{N}}$ and, for each integer $n \in [1,\ M]$, a sequence $\{F_{\nu,\ n}\}_{\nu\in\mathbf{N}}$ of functions holomorphic in $\mathcal{Z}_n$, such that, for every $n$ and $m$ in $[1,\ M]$, $z \in \mathcal{Z}_n$, $w \in \mathcal{Z}_m^*$,*

$$\mathrm{W}_{m+n}(w_{\leftarrow},\ z) = \sum_{\nu\in\mathbf{N}} \overline{F_{\nu,\ m}(\overline{w})}\, F_{\nu,\ n}(z), \tag{113}$$

*where the convergence is uniform on every compact subset of $\mathcal{Z}_m^* \times \mathcal{Z}_n$.*

The domain $U_1$ of Eq. (139) is not a tube. We can however inscribe in it increasing unions of topological products of lunules which are isomorphic to tubes. In fact, for every $A > \frac{1}{\pi}$, there exists an $\varepsilon > 0$ such that $U_1$ contains

$$\begin{aligned} V_A = \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in L(A,\ \varepsilon),\quad \zeta_j \in L(A,\ \pi - 1/A)\} \\ \cup\ \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ \omega \in L(A,\ \pi - 1/A),\quad \zeta_j \in L(A,\ \varepsilon)\}. \end{aligned} \tag{141}$$

Using the conformal map (128) in all variables, we can map $V_A$ into a tube whose holomorphy envelope is its convex hull. Returning to the variables $(\omega,\ \zeta)$, and taking the limit $A \to \infty$ shows that the functions $G_{s,a}$ are holomorphic in the interior of the convex hull of $S_1$, namely

$$\bigcup_{0<\theta<\pi} \{(\omega,\ \zeta) \in \mathbf{C}^{1+N}\ :\ 0 < \operatorname{Im}\omega < \pi - \theta,\quad 0 < \operatorname{Im}\zeta_j < \theta,\ \forall j\}. \tag{142}$$

This set is the image of the domain $\Delta_a$ introduced in Eq. (124) under the mapping $w \mapsto \mu = w - w^{-1} \mapsto \omega$, $z \mapsto \zeta$ defined in Eq. (135), and therefore the assertion of Lemma 4 follows.

Lemma 3 (i) in the special case $D = P$ follows from the latter by letting $a$ tend to 0.
2. We now prove Lemma 3 (ii) in the case when $D' = \rho\, Q$ for some real $\rho > 1$. The proof of this is the same as that of Lemma 4, except that the change of coordinates (138) is replaced by

$$z_j = \exp(i\zeta_j), \quad (1 \leq j \leq N). \tag{143}$$

This again allows the use of the tube theorem.
3. Lemma 3 (ii) follows from this by using chains of polydiscs, and (i) follows in the same way from the special case $D = P$ and (ii).



# B Appendix: A lemma of Hall and Wightman

In [23], Hall and Wightman proved the following lemma

**Lemma 5** *Let $M \in L_+(\mathbf{C})$ be such that $\mathrm{T}_+ \cap M^{-1}\,\mathrm{T}_+ \neq \emptyset$. There exists a continuous path $t \mapsto M(t)$ from the interval $[0,\ 1]$ into $L_+(\mathbf{C})$ such that $M(0) = 1$, $M(1) = M$ and that, for every $z \in \mathrm{T}_+ \cap M^{-1}\,\mathrm{T}_+ \subset \mathbf{C}^{d+1}$, $M(t)z \in \mathrm{T}_+$ holds for all $t \in [0,\ 1]$.*

This lemma is proved in [23] for the case $d + 1 \leq 4$ (a very clear exposition also appears in [32]). It is extended to all dimensions in [24]. We give another proof based on holomorphic continuation. As noted in the above references, if $M \in L_+(\mathbf{C})$ is such that the statement in Lemma 5 holds, then it holds for $\Lambda_1 M \Lambda_2$ for any $\Lambda_1,\ \Lambda_2 \in L_+^\uparrow$, as well as for $M^{-1}$. It is therefore sufficient to consider the case when $M$ is one of the normal forms classified by Jost in [24]. $M$ can then be written in the form:

$$M = \begin{pmatrix} M_1(i) & 0 \\ 0 & M_2(i) \end{pmatrix} \tag{144}$$

where $t \mapsto M_1(t)$ is a one-parameter subgroup of the $p \times p$ Lorentz group, real for real $t$, with $p \leq 3$, and $t \mapsto M_2(t)$ is a one-parameter subgroup of the $(d+1-p) \times (d+1-p)$ orthogonal group, real for real $t$. In the generic case $p \leq 2$, $M_1(t) = 1$ if $p = 1$ and, if $p = 2$, $M_1(t) = [\exp at]$ for some real $a$ with $|a| \leq \pi$. We focus on this case first. Replacing $M$ by $M^{-1}$ if necessary, we may assume $0 < a \leq \pi$. For any $z \in \mathrm{T}_+$ the set $\Delta(z,\ M) = \{t \in \mathbf{C}\ :\ M(t)z \in \mathrm{T}_+\}$ is invariant under real translations, i.e. is a union of open strips parallel to the real axis. Let

$$E(M) = \{\mathrm{T}_+ \cap M^{-1}\mathrm{T}_+\} = \{z \in \mathbf{C}^{d+1}\ :\ \mathbf{R} \cup (i + \mathbf{R}) \subset \Delta(z,\ M)\}.$$

Denote $z(s) = (z^{(0)},\ z^{(1)},\ sz^{(2)},\ \ldots,\ sz^{(d)})$. If $z \in E(M)$, then $z(s) \in E(M)$ for all $s \in [0,\ 1]$. The set $\Delta(z(0),\ M)$ contains the segment $i[0,\ 1]$, and hence $i[0,\ 1] \subset \Delta(z',\ M)$ for all $z'$ in a sufficiently

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