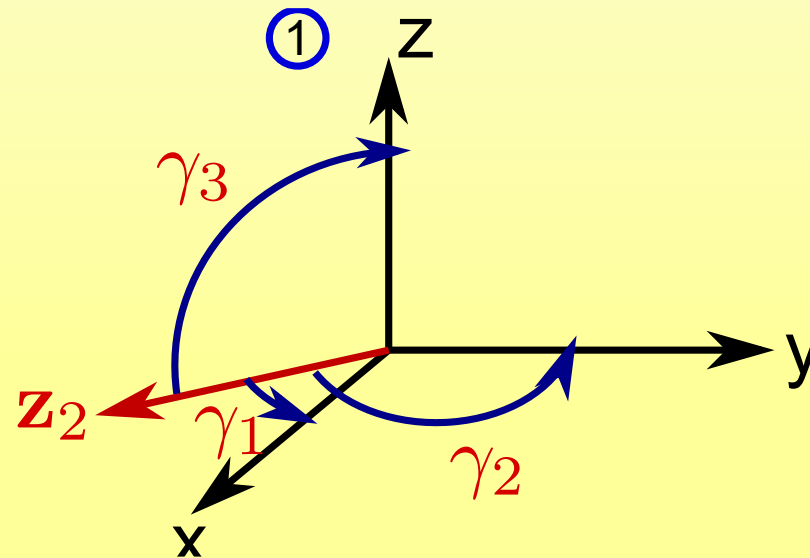
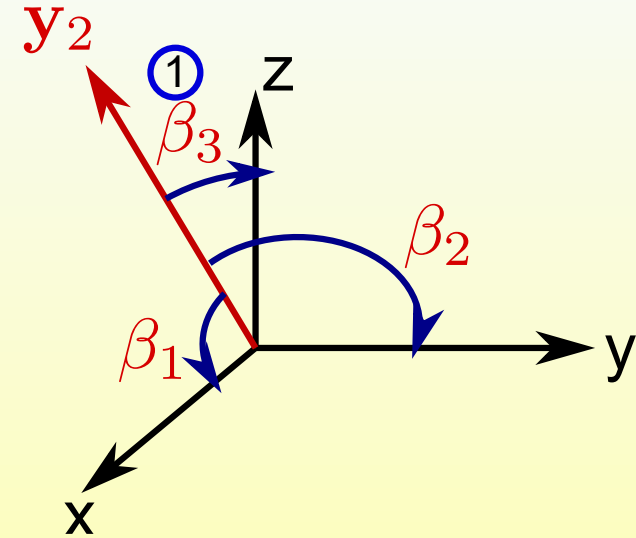
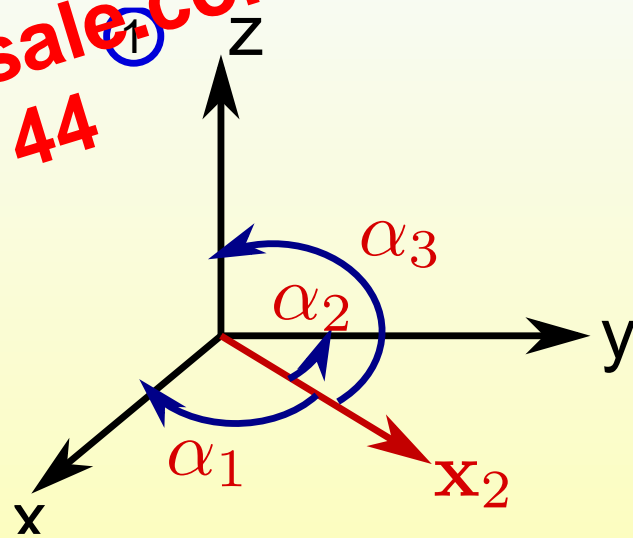


# Rotation matrix representation of orientation

- $\hat{\mathbf{x}}_2^1 = \begin{bmatrix} \cos \alpha_1 \\ \cos \alpha_2 \\ \cos \alpha_3 \end{bmatrix}$
- $\hat{\mathbf{y}}_2^1 = \begin{bmatrix} \cos \beta_1 \\ \cos \beta_2 \\ \cos \beta_3 \end{bmatrix}$
- $\hat{\mathbf{z}}_2^1 = \begin{bmatrix} \cos \gamma_1 \\ \cos \gamma_2 \\ \cos \gamma_3 \end{bmatrix}$

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# Rotation matrix applications: vector transformation

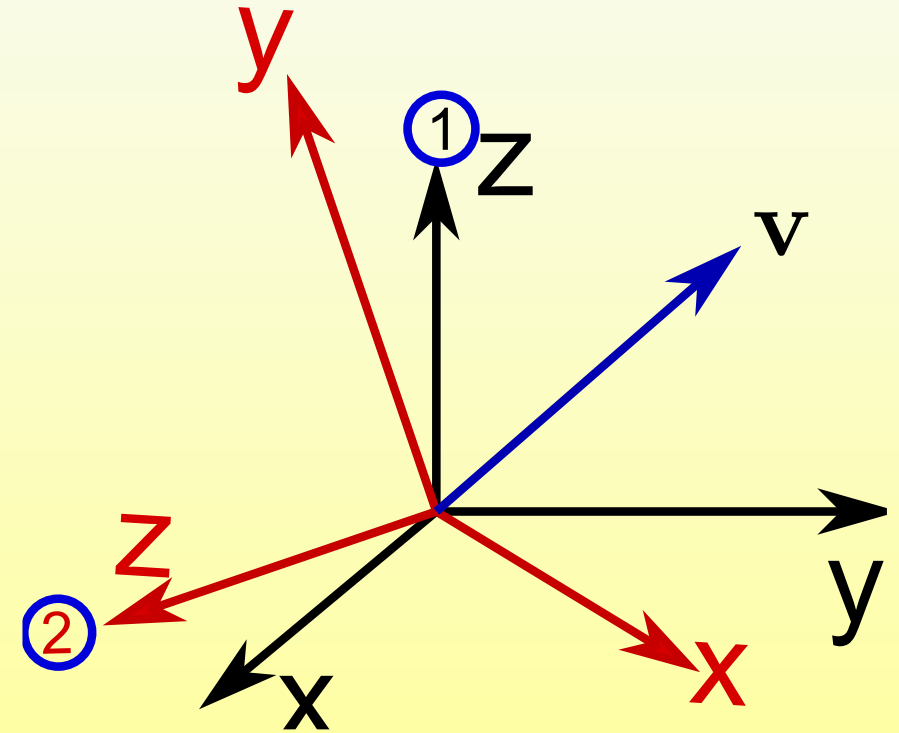
- $\mathbf{v}^1 = v_1^2 \mathbf{x}_2^1 + v_2^2 \mathbf{y}_2^1 + v_3^2 \mathbf{z}_2^1$

- $\rightarrow \mathbf{v}^1 = \begin{bmatrix} \mathbf{x}_2^1 & \mathbf{y}_2^1 & \mathbf{z}_2^1 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix}$

- But the rotation matrix  $\mathbf{R}_2^1 = \begin{bmatrix} \mathbf{x}_2^1 & \mathbf{y}_2^1 & \mathbf{z}_2^1 \end{bmatrix}$

- $\rightarrow \mathbf{v}^1 = \mathbf{R}_2^1 \mathbf{v}^2$

- The rotation matrix can be used to transform a vector from being expressed using frame 2 coordinates to being expressed using frame 1 coordinates.



# Properties of rotation matrix

2. Given that  $\mathbf{R}_2^1$  is the rotation matrix that describes the rotation from frame 1 to frame 2 and  $\mathbf{R}_1^2$  is the rotation matrix that describes the rotation from frame 2 to frame 1, then:

- $\mathbf{R}_1^2 = \mathbf{R}_2^1{}^T$

- Proof:

- Let  $\mathbf{v}$  be a vector that is expressed in frame 1 by  $\mathbf{v}^1$  and in frame 2 by  $\mathbf{v}^2$ .

- $\mathbf{v}^1 = \mathbf{R}_2^1 \mathbf{v}^2$

- $\rightarrow \mathbf{v}^2 = \mathbf{R}_2^1{}^{-1} \mathbf{v}^1$

- By the definition of  $\mathbf{R}_1^2$ , we know that:  $\mathbf{v}^2 = \mathbf{R}_1^2 \mathbf{v}^1$

- $\therefore \mathbf{R}_1^2 = \mathbf{R}_2^1{}^{-1} = \mathbf{R}_2^1{}^T$

Proof that any rotation matrix has at least one eigen value  $\lambda = 1$

• Eigen values must satisfy the characteristic equation of the matrix  $\mathbf{R}_2^1$

•  $\det(\lambda \mathbf{I} - \mathbf{R}_2^1) = 0$

• For  $\lambda = 1$ , we have:

•  $\det(\mathbf{I} - \mathbf{R}_2^1) = \det\left(\mathbf{R}_2^1 \left(\mathbf{R}_2^{1T} - \mathbf{I}\right)\right) = \det(\mathbf{R}_2^1) \det\left(\mathbf{R}_2^{1T} - \mathbf{I}\right)$

•  $\rightarrow \det(\mathbf{I} - \mathbf{R}_2^1) = \det\left(\mathbf{R}_2^{1T} - \mathbf{I}\right) = \det(\mathbf{R}_2^1 - \mathbf{I}) = -\det(\mathbf{I} - \mathbf{R}_2^1)$

•  $\rightarrow 2 \det(\mathbf{I} - \mathbf{R}_2^1) = 0$

•  $\rightarrow \det(\mathbf{I} - \mathbf{R}_2^1) = 0$

• Hence  $\lambda = 1$  is an eigen value of the rotation matrix  $\mathbf{R}_2^1$

# Properties of rotation matrix

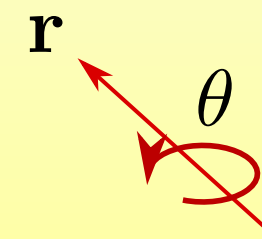
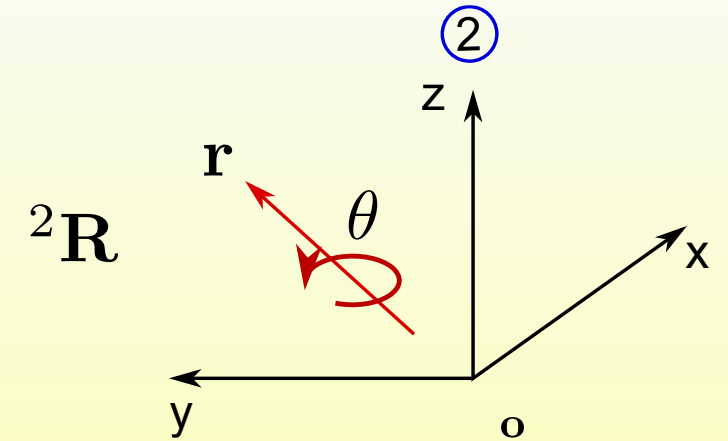
7. For any rotation matrix  $\mathbf{R}_2^1$  and any two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , the inner product of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is equal to the inner product of the two rotated vectors  $\mathbf{R}_2^1 \mathbf{v}_1$  and  $\mathbf{R}_2^1 \mathbf{v}_2$ . i.e  $\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{R}_2^1 \mathbf{v}_1) \cdot (\mathbf{R}_2^1 \mathbf{v}_2)$

• Proof:

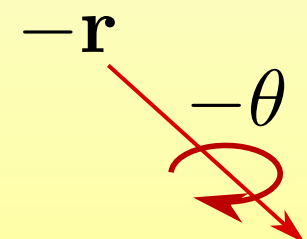
$$\bullet (\mathbf{R}_2^1 \mathbf{v}_1) \cdot (\mathbf{R}_2^1 \mathbf{v}_2) = (\mathbf{R}_2^1 \mathbf{v}_1)^T (\mathbf{R}_2^1 \mathbf{v}_2) = \mathbf{v}_1^T \mathbf{R}_2^{1T} \mathbf{R}_2^1 \mathbf{v}_2 = \mathbf{v}_1^T \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2$$

# Axis of rotation and angle of rotation

- Any rigid rotation can be represented as a rotation by an angle  $\theta$  around an axis  $\mathbf{r}$
- There are two solutions for the angle of rotation and axis of rotation
- The axis of rotation  $\mathbf{r}$  is the vector that doesn't change direction with rotation, i.e.  ${}^2\mathbf{R}\mathbf{r} = \mathbf{r}$ .
  - Hence,  $\mathbf{r}$  can be determined by solving the equation  $({}^2\mathbf{R} - \mathbf{I})\mathbf{r} = \mathbf{0}$  with the constraint that  $\|\mathbf{r}\| = 1$
- What is the rotation angle  $\theta$ ?



First solution



Second solution

# Rotation axis and angle of $\mathbf{R}^T$

- Consider a rotation  $\mathbf{R}$  with rotation axis  $\mathbf{r}$  and rotation angle  $\theta$
- The rotation  $\mathbf{R}^T$  has the same rotation axis  $\mathbf{r}$  but with rotation angle  $-\theta$
- Proof:
  - Given that:  $\mathbf{R}\mathbf{r} = \mathbf{r}$
  - $\Rightarrow \mathbf{r} = \mathbf{R}^T\mathbf{r} \Rightarrow \mathbf{R}^T\mathbf{r} = \mathbf{r}$
  - Hence,  $\mathbf{r}$  is the rotation axis of  $\mathbf{R}^T$ .
  - We also have that:  $tr(\mathbf{R}^T) = tr(\mathbf{R}) = 1 + 2 \cos(\theta)$
  - $\Rightarrow \mathbf{R}^T$  and  $\mathbf{R}$  have the same  $\cos(\theta)$
  - It can be shown that  $\mathbf{R}^T$  has a rotation angle of  $-\theta$

