16.1.6 Sketching the graphs

Sketching A Parabola

We need to consider the two types of possible parabolas, which are vertical parabolas and horizontal parabolas.

Sketching A Vertical Parabola

To graph a vertical parabola:

- Locate and label the vertex (h, k) through finding the focus and then using the distance property we discussed earlier in that the vertex is p units from the focus

- Graph the axis of symmetry $x = h$, with a dashed line

- Graph enough points to see a pattern. The $x -$ and $y -$ intercepts are important points to determine.

- Connect the points with a smooth curve

Sketching A Horizontal Parabola

To graph a horizontal parabola:

- Locate and label the vertex (h, k) through finding the focus and then using the distance property we discussed earlier in that the vertex is p units from the focus

- Graph the axis of symmetry $y = k$, with a dashed line

- Graph enough points to see a pattern. The $x -$ and $y -$ intercepts are important points to determine.

- Connect the points with a smooth curve

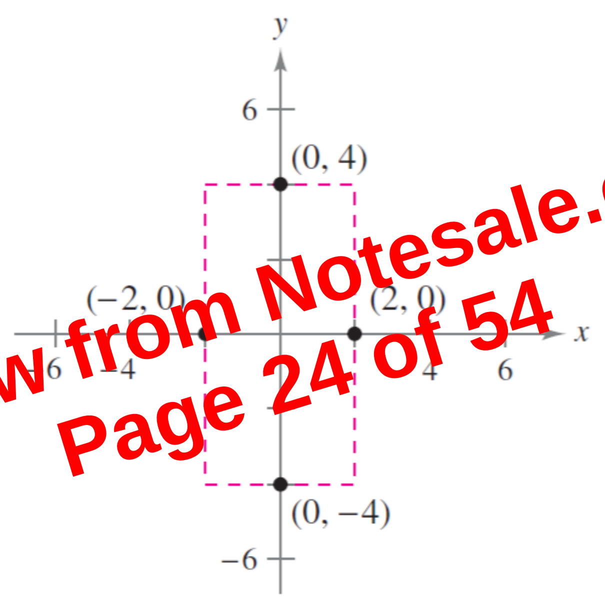
16.1.6 Sketching the graphs

Sketching A Hyperbola

We shall use an example, which can be generalised to any sort of questions regarding the sketching of hyperbolas.

Sketch the hyperbola given by $4x^2 - y^2 = 16$.

The first step is writing this equation in standard form like was shown on the previous page. We ultimately get $\frac{x^2}{2^2} - \frac{y^2}{4^2} = \mathbf{1}$ and because the x^2 − term is positive, you can conclude that the **transverse axis** is **horizontal**. The vertices occur at $(-2,0)$ and $(2,0)$. Moreover, the endpoints of the conjugate axis occur at $(0,-4)$ and $(0,4)$. With this information, you can sketch the rectangle shown below.



16.1.6 Sketching the graphs

Completing The Square: Vertical Parabola

This form of completing the square we already know. Given the equation of a vertical parabola of the form $x^2 + y - 24x + 10 = 0$ (we can tell it is a vertical parabola because the squared term is x) we proceed as follows:

Since we are dealing with a vertical parabola, make y the subject of the formula	$\mathbf{y = -x^2 + 24x - 10}$
Factorise the terms in x and x^2 in the equation so as to obtain an equation where the term in x^2 has a coefficient of + 1 **Divide the second degree term (term in x^2) and the first degree term (term in x) by x and square the resultant set of brackets**	$y = -(x^2 - 24x) - 10$ $y = -\left(\frac{\mathbf{x^2}}{\mathbf{x}} - \frac{\mathbf{24x}}{\mathbf{x}}\right) - 10$ $y = -(\mathbf{x} - \mathbf{24})^2 - 10$
Leave space towards the right end of the equation: + ( ) **Take half of the constant within the first set of brackets and do not forget its sign then proceed to replace the original constant with half its value, square this half value, multiply it by the value of the factor on the squared bracket and add the square to the space towards the right end of the equation**	$y = -(x - \mathbf{24})^2 - 10 + (\quad)$ $\frac{\mathbf{1}}{\mathbf{2}} \times (\mathbf{-24}) = \mathbf{-12}$ **then** $\mathbf{1} \times (\mathbf{-12})^2 = \mathbf{144}$ $y = -(x - \mathbf{12})^2 - 10 + (\mathbf{144})$

It is **important** to note that whenever the coefficient of the square bracket is negative we **must** leave space towards the right end of the equation in the form $+ (\quad)$ that way we **add the square of half the value of the constant**. However, if the coefficient of the square bracket is positive then we **must** leave space towards the right end of the equation in the form $- (\quad)$ that way we **subtract the square of half the value of the constant**. The reason for this is evident once you **expand the completed square form**.

16.1.6 Sketching the graphs

This ultimately gives us the completed square form which when expanded can be seen to give the original equation we started with:

$$\mathbf{x = -\frac{1}{2}(y+16)^2 + 124 \text{ which is the same as } x = -\frac{y^2}{2} - 16y - 4}$$

We can now obtain the vertex of the graph of this parabola:

$$\mathbf{\text{First, we make } -\frac{1}{2}(y+16)^2 = 0 \text{ and solve for x } \therefore x = 0 + 124 \text{ and so } x = 124}$$

$$\mathbf{\text{Then, using } -\frac{1}{2}(y+16)^2 = 0 \text{ we solve for y } \therefore -y - 16 = 0 \text{ and so } y = -16}$$

We therefore get the vertex of the graph of this parabola to be $\mathbf{(124, -16)}$ which we can now use to sketch the graph.

16.1.6 Sketching the graphs

It is **important** to note that when dealing with both kinds of hyperbolae (horizontal and vertical) we **must** leave the spaces on the right side of the equation as mentioned in the table on the previous page in the form $+\frac{a}{b}(\ \ )$ or $-\frac{c}{d}(\ \ )$ depending on whether **the factor from the respective bracket is positive or negative** that way we **add or subtract the square of half the coefficient of the first degree terms** $(\textbf{terms in x and y})$ whereas we **must** leave the spaces on the left side of the equation as mentioned in the table on the previous page in the form $+(\ \ )$ that way we **add the square of half the coefficient of the first degree terms** $(\textbf{terms in x and y})$.

From our table, after simplifying the quadratic equations on the left hand side of the equation, finding the difference of the values on the right hand side of the equation and dividing the whole equation through by the total of this difference we get the completed square form which when expanded can be seen to give the original equation we started with:

$$\frac{(\mathbf{y}+\mathbf{2})^{\mathbf{2}}}{\mathbf{3}^{\mathbf{2}}}-\frac{(\mathbf{x}-\mathbf{1})^{\mathbf{2}}}{\mathbf{4}^{\mathbf{2}}}=\mathbf{1}\ \textbf{which is the same as}\ -9x^2+18x+16y^2+64y=89$$

We can now obtain the centre of the graph of this hyperbola:

$$\textbf{First, we make } (\mathbf{y}+\mathbf{2})^{\mathbf{2}}=\mathbf{0}\ \textbf{and solve for y} \therefore \mathbf{y}+\mathbf{2}=\mathbf{0}\ \textbf{and so}\ \mathbf{y}=-\mathbf{2}$$

$$\textbf{Then, using } (\mathbf{x}-\mathbf{1})^{\mathbf{2}}=\mathbf{0}\ \textbf{we solve for x} \therefore \mathbf{x}-\mathbf{1}=\mathbf{0}\ \textbf{and so}\ \mathbf{x}=\mathbf{1}$$

We therefore get the centre of the graph of this hyperbola to be $(\mathbf{1},-\mathbf{2})$ which we can now use to obtain the vertices as from what we have learnt and subsequently sketch the graph.

16.1.7 Finding points of intersection with the coordinate axes or other straight lines. Candidates will be expected to interpret the geometrical implication of equal roots, distinct real roots or no real roots

It is common practice to find these points of intersection/s before sketching the graphs as it will make it easier to already know what the graph looks like before actually sketching it.

Coordinate Axes

This refers to the $\mathrm{x-axis\ and\ y-axis}$.

To find the point of intersection between a rational function and the $\mathrm{x-axis}$ you simply have to evaluate $\mathrm{y}=0$ and obtain the coordinate/s in the form $(\mathrm{x}, 0)$.

To find the point of intersection between a rational function and the $\mathrm{y-axis}$ you simply have to evaluate $\mathrm{x}=0$ and obtain the coordinate/s in the form $(0, \mathrm{y})$.

Other Straight Lines

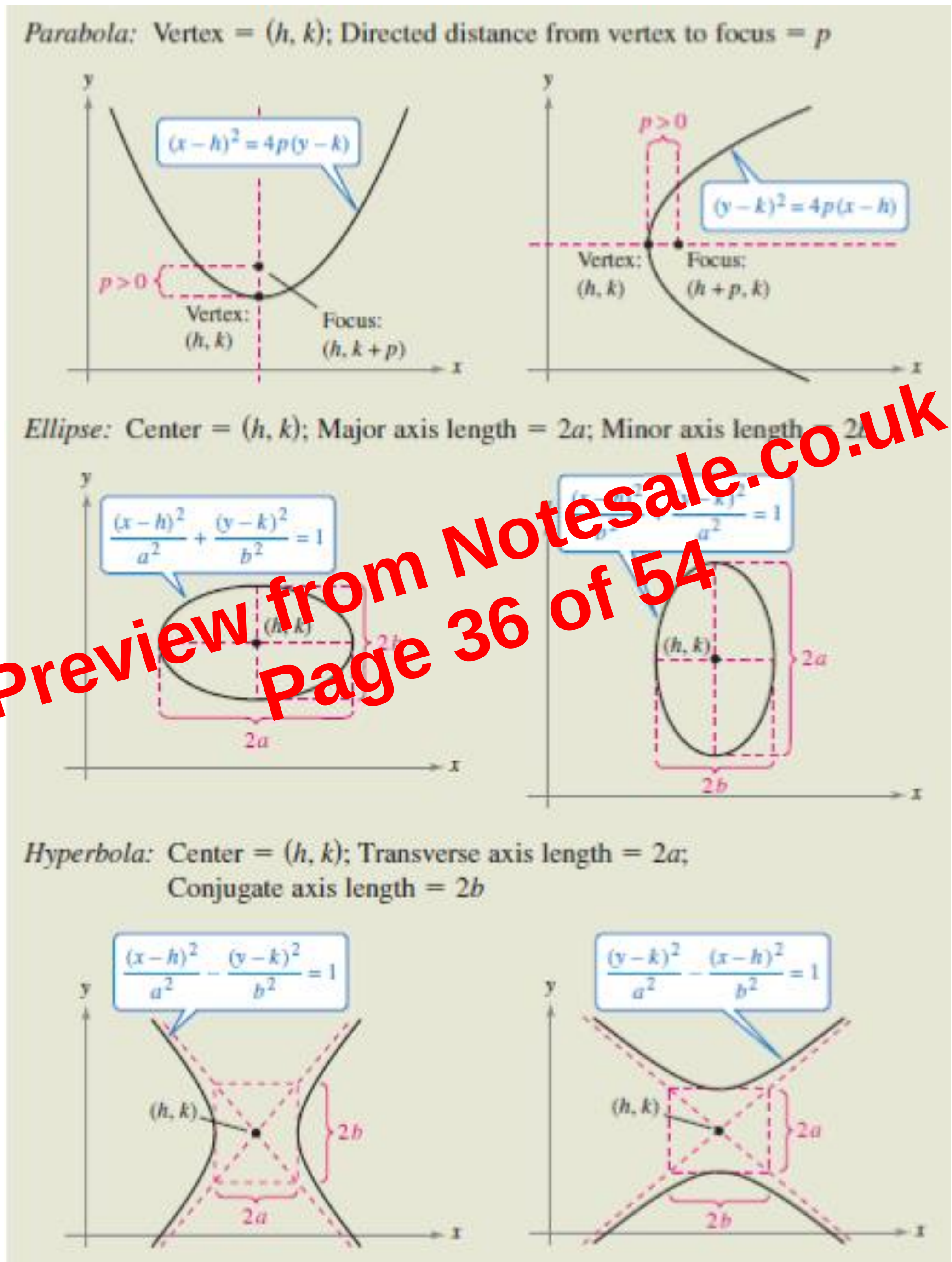
This refers to any other lines other than the rational function given.

To find the point of intersection between a rational function and $\mathrm{any\ other\ line}$ you need to equate their equations and solve $\mathrm{first\ for\ x}$ then $\mathrm{for\ y}$ and obtain the coordinate/s in the form (x, y).

Sketching and identifying the points of intersection/s follows the same manner as that of graphs of rational functions.

16.1.8 Knowledge of the effects on these equations of single transformations of these graphs involving translations, stretches parallel to the x – axis or y – axis, and reflections in the line y = x. Including the use of the equations of the asymptotes of the hyperbolas given in the formulae booklet

The following summary lists the standard forms of the equations of the four basic conics.



16.1.8 Knowledge of the effects on these equations of single transformations of these graphs involving translations, stretches parallel to the $x-$ axis or $y-$ axis, and reflections in the line $y = x$. Including the use of the equations of the asymptotes of the hyperbolas given in the formulae booklet

Effect On The Equation Of A Conic After Being Stretched Vertically Or Horizontally Or Both

Vertical Parabola

A vertical parabola has an equation of the form $x^2 = 4py$ and usually $4p = 1$ to give the simple parabola $y = x^2$ however, it is also possible to stretch a vertical parabola. The stretching occurs such that the vertical stretch of the parabola is inversely proportional to its horizontal stretch, which essentially means as the parabola stretches vertically it compresses horizontally and vice versa. These changes occur simultaneously on every vertical parabola and as such, changes horizontally must be accompanied by changes vertically.

As with every stretch, there is a stretch scale factor for every vertical parabola. We shall expound on how we can obtain the value of this scale factor and be able to recognise it on the equation of a parabola.

Suppose we start off with the equation of a vertical parabola $y = x^2$ then we multiply this parabola by a factor of $\mathbf{a}$ we then obtain the following equation:

$$y = ax^2$$

If $\mathbf{a = 1}$ this is the same equation as before. Let us consider some other possibilities:

- If $\mathbf{a} > 1$ we increase the value of $\mathbf{y}$ and decrease the value of $\mathbf{x}$ so we **stretch the graph horizontally** and **compress the graph vertically**

- If $\mathbf{0} < a < 1$ we decrease the value of $\mathbf{y}$ and increase the value of $\mathbf{x}$ so we **compress the graph horizontally** and **stretch the graph vertically**

- If $\mathbf{a = -1}$ we replace $\mathbf{y}$ with $\mathbf{-y}$ which geometrically results in **a reflection of the graph of the parabola through the $\mathbf{y-}$ axis or the line $\mathbf{x = 0}$**

- If $\mathbf{-1} < a < 0$, in the now reflected parabola, we decrease the value of $\mathbf{y}$ and increase the value of $\mathbf{x}$ so we **compress the graph horizontally** and **stretch the graph vertically**

- If $\mathbf{a} < 1$, in the now reflected parabola, we increase the value of $\mathbf{y}$ and decrease the value of $\mathbf{x}$ so we **stretch the graph horizontally** and **compress the graph vertically**

Ultimately, the **stretch scale factor** is given by the **value of $\mathbf{a}$**.

16.1.8 Knowledge of the effects on these equations of single transformations of these graphs involving translations, stretches parallel to the $x-$axis or $y-$axis, and reflections in the line $y=x$. Including the use of the equations of the asymptotes of the hyperbolas given in the formulae booklet

Horizontal Ellipse (Major Axis: Horizontal)

Now we will look at all possible stretches that can be done on a horizontal ellipse:

- If $\mathbf{a^2}$ **becomes** $\mathbf{(2a)^2}$ such that the new equation of the ellipse becomes $\frac{x^2}{(2a)^2}+\frac{y^2}{b^2}$ will result in a **stretch of the graph horizontally by a factor of 2** and **no change in the graph vertically**; therefore if these inequalities are true: **coefficient of a** >1 or coefficient of $a<-1$ you have a **horizontal stretch** with a **scale factor** that is the **coefficient of a** and **no change vertically**

- If $\mathbf{a^2}$ **becomes** $\left(\frac{1}{2}a\right)^2$ such that the new equation of the ellipse becomes $\frac{x^2}{\left(\frac{1}{2}a\right)^2}+\frac{y^2}{b^2}$ will result in a **compression of the graph horizontally by a factor of** $\frac{1}{2}$ and **no change in the graph vertically**; therefore if this inequality is true: $-1<$ coefficient of $a<1$ you have a **horizontal compression** with a **scale factor** that is the **coefficient of a** and **no change vertically**

- If $\mathbf{b^2}$ **becomes** $\mathbf{(6b)^2}$ such that the new equation of the ellipse becomes $\frac{x^2}{a^2}+\frac{y^2}{(6b)^2}$ will result in a **stretch of the graph vertically by a factor of 6** and **no change in the graph horizontally**; therefore if these inequalities are true: **coefficient of b** >1 or coefficient of $b<-1$ you have a **vertical stretch** with a **scale factor** that is the **coefficient of b** and **no change horizontally**

- If $\mathbf{b^2}$ **becomes** $\left(\frac{3}{4}b\right)^2$ such that the new equation of the ellipse becomes $\frac{x^2}{a^2}+\frac{y^2}{\left(\frac{3}{4}b\right)^2}$ will result in a **compression of the graph vertically by a factor of** $\frac{3}{4}$ and **no change in the graph horizontally**; therefore if this inequality is true: $-1<$ coefficient of $b<1$ you have a **vertical compression** with a **scale factor** that is the **coefficient of b** and **no change horizontally**

- If **a and b** both change then the ellipse is either **stretched or compressed or both** with the **scale factors** that are the **coefficients of a and b** however in different directions such that **a affects the ellipse horizontally and b affects the ellipse vertically**