# *Calculus Notes*

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continuous at $b$.

The **intermediate value theorem** says that for a function $f$ continuous on $[a, b]$, for any $y_0$ between $f(a)$ and $f(b)$, $y_0 = f(c)$ for some $c \in [a, b]$. It can be used to show the existence of roots to an equation within an interval.

Some functions $f$ may not be defined at some $c$, yet $\lim_{x\to c} f(x) = L$ might still exist. Then, a **continuous extension** of $f$ can be defined by

$$F(x) = \begin{cases} f(x) \text{ if } x \in D_f \\ L \text{ if } x = c \end{cases}$$

## 2.5 Limits involving infinity and asymptotes

We say that $f$ has a limit $L$ as $x$ approaches infinity and write $\lim_{x\to\infty} f(x) = L$ if for all $\epsilon > 0$, there is an $M \in \mathbb{R}$ for which

$$|f(x) - L| < \epsilon \text{ whenever } x > M$$

Similarly, $\lim_{x\to-\infty} f(x) = L$ if for all $\epsilon > 0$, there is an $N \in \mathbb{R}$ for which

$$|f(x) - L| < \epsilon \text{ whenever } x < N$$

The limit laws from §2.1 apply here.



A line $y = b$ is a **horizontal asymptote** to the graph $y = f(x)$ if $\lim_{x\to-\infty} f(x) = b$ or $\lim_{x\to\infty} f(x) = b$.

Rational functions where the numerator has a degree 1 larger than the denominator have an **oblique asymptote** obtained through polynomial long division.

**Infinite limits** are ways to describe function behaviour near points of infinite discontinuity. We say $f$ approaches infinity as $x \to c$, and write $\lim_{x\to c} f(x) = \infty$ if for all $B > 0$, there exists $\delta > 0$ for which

$$f(x) > B \text{ whenever } 0 < |x - c| < \delta$$

We say $f$ approaches negative infinity as $x \to c$, and write $\lim_{x\to c} f(x) = -\infty$ if for all $B > 0$, there exists $\delta > 0$ for which

$$f(x) < -B \text{ whenever } 0 < |x - c| < \delta$$

A line $x = a$ is a **vertical asymptote** of the graph $y = f(x)$ if $\lim_{x\to a^-} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = \infty$.

The right-hand derivative is defined similarly. For $f$ to be differentiable on $[a, b]$, it must be differentiable on $(a, b)$, right-differentiable at $a$ and left-differentiable at $b$.

Differentiability implies continuity. That is, if $f$ is differentiable at $c$, it is necessary that it is continuous there as well. However, the opposite might not be true. A continuous function can fail to be differentiable due to a corner, cusp or a vertical tangent line, amongst other reasons.

## 3.3 Differentiation rules

Using the definition in §3.2, we can prove various rules for differentiation. The **power rule** states that for $n \in \mathbb{R}$,

$$\frac{d}{dx}x^n = nx^{n-1}$$

The sum and constant multiple rules allow us to take derivatives of constant multiples and sums of functions. For the remainder of this section, let $u, v$ be functions and $c \in \mathbb{R}$.

$$\frac{d}{dx}(cu) = c\frac{du}{dx} \text{ and } \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

The **product rule** says that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

Note that this can be extended to multiple functions to evaluate $(y_1 y_2 ... y_n)'$.

The **quotient rule** says that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

If $f'$ itself is a differentiable function, then we can take the **second-order derivative** $f'' = (f')'$. In general, the **nth-order derivative** $f^{(n)}$ is found from differentiating the previous derivative: $f^{(n)} = (f^{(n-1)})'$.

## 3.4 The derivative as a rate of change

By interpreting derivatives as rates, we can use calculus to analyse motion. Consider a particle moving obeying the **position** function $s$. A **displacement** is a change in position $\Delta s$.

The **velocity** $v = \frac{ds}{dt}$ is the instantaneous rate of change of position. The **speed** is the magnitude of velocity $|v|$.

In §5.1, we discussed areas. We can define the **area under a curve** $f(x)$ over $[a, b]$ for a nonnegative $f$ as

$$A = \int_a^b f(x)\,dx$$

The **mean** of $f$ over $[a, b]$ is defined as

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

## 5.4 The fundamental theorem of calculus

The fundamental theorem of calculus links the definite integral to the derivative. This lets us evaluate integrals without taking limits of Riemann sums.

The **mean value theorem for definite integrals** says that for a function $f$ continuous on $[a, b]$, there exists some $c \in [a, b]$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The **first part of the fundamental theorem of calculus** involves a function $F(x) = \int_a^x f(t)\,dt$ which is continuous on $[a, b]$ and differentiable on $(a, b)$ when $f$ is continuous on $(a, b)$. The theorem then states that $F$ is an antiderivative of $f$. That is,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



The **second part of the fundamental theorem of calculus (evaluation theorem)** gives that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thus, to evaluate an integral, we need only find an antiderivative and evaluate it at the endpoints of the interval.

Writing $f = F'$ in the above, we get the **net change theorem**, which says that the integral of a rate over an interval is the net change of the function over that interval.

The two parts of the fundamental theorem essentially show that differentiation and integration are "inverse" operations.

Revisiting areas for more general functions that can be both positive and negative, we need to break up the interval into subintervals for which the function is either entirely positive or negative and perform the integrals separately. The

## 7.4 Exponential change and separable differential equations

Many real-life phenomena obey the property that their rate of change is proportional to the quantity. This yields the initial value problem:

$$\frac{dy}{dt} = ky, y(0) = y_0$$

The solution to this is $y = y_0 e^{kt}$, and the quantity is said to undergo **exponential change**. If $k > 0$, the quantity undergoes **exponential growth**. If $k < 0$, it undergoes **exponential decay**. $k$ is the **rate constant**.

The above differential equation is an example of a **separable** differential equation, which takes the form

$$\frac{dy}{dx} = g(x)h(y)$$

We can write this in differential form and integrate to get the implicit solution

$$\int \frac{dy}{h(y)}\, dy = \int g(x)\, dx$$

Exponential functions are good models for many real situations. If population growth is unrestricted, we can model it with exponential growth.

Radioactive decay follows exponential decay. The **half-life** is the time required for a sample of radioactive material to decay to half its initial mass. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}$$



Heat transfer also obeys exponential decay. Newton's law of cooling says that the temperature difference $H - H_S$, with $H_S$ being the temperature of the surroundings, is directly proportion to the initial temperature difference.

## 7.5 Indeterminate forms and L'Hôpital's rule

An expression is in **indeterminate form** if its value cannot be assigned without further information. Examples of indeterminate forms include $0/0, \infty/\infty, 0 \cdot \infty, \infty - \infty, \infty^0$ and $1^\infty$.

**Cauchy's mean value theorem** stipulates that for $f, g$ continuous on $[a, b]$ and differentiable on $(a, b)$, with $g'(x) \neq 0$ throughout $(a, b)$, then there exists $c \in (a, b)$ so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

## 8.7 Numerical integration

We may seek a numerical approximation to a definite integral $\int_a^b f(x)\,dx$ if it is hard or impossible to find an antiderivative of $f$. The Riemann sum (§5.2) is an example of an approximation. We now discuss the trapezoidal rule and Simpson's rule.

For the sake of convenience, we assume $n$ equally-spaced subintervals with $\Delta x = (b-a)/n$. $\Delta x$ is called the **step size**.

The **trapezoidal rule** involves using areas under trapezoids in each subinterval. The formula for each trapezoid's area is $\Delta x(y_{k-1}+y_k)/2$, so the approximation is given by

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + ... + 2y_{n-1} + y_n)$$

**Simpson's rule** involves using areas under parabolas. Each parabolic piece is placed over 2 subintervals, so we need $n$ to be even. The formula for each parabolic region's area is $\Delta x(y_{k-1} + 4y_k + y_{k+1})/3$, so the approximation is given by

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + ... + 2y_{n-2} + 4y_{n-1} + y_n)$$

It is useful to find the maximum error from using the above approximations. The **error** is the difference between the approximated and actual value of the integral. We have

$$|E_T| \le \frac{M(b-a)^3}{12n^2} \text{ and } |E_S| \le \frac{M(b-a)^5}{180n^4}$$

In the formula for $|E_T|$, we need $f''$ to be continuous. Then, $M$ is an upper bound of $\max|f''|$ on $[a,b]$. Similarly, in the formula for $|E_S|$, we need $f^{(4)}$ to be continuous. Then, $M$ is an upper bound of $\max|f^{(4)}|$ on $[a,b]$. In many cases, finding the maximum of these derivatives exactly is not possible and we have to find an upper bound.

Notice that the error for the trapezoidal rule varies with the inverse square of $n$, whereas the Simpson's rule varies with the inverse fourth power of $n$. Thus, Simpson's rule gives a better approximation. Also, the error formulae show that Simpson's rule gives the exact value of the integral for constant, linear, quadratic and cubic functions. Similarly, the trapezoidal rule gives the exact value for constant and linear functions.

## 8.8 Improper integrals

We can extend our definition of integrals to infinite intervals and functions that have vertical asymptotes in their interval of integration.

We define a **random variable** $X$ to be a function assigning a probability to each outcome in a sample space. If there are finitely many outcomes, $X$ is **discrete**. Else, it is **continuous**.

For continuous random variables, we define their **probability distribution function** $f$ to have the properties that $f$ is defined on $\mathbb{R}$, has finitely many discontinuities, is nonnegative and is such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Then, we define the **probability** that $X$ takes on a value between $c$ and $d$ to be

$$\mathrm{P}(c \leq X \leq d) = \int_c^d f(x)\,dx$$

The **median** $m$ of $X$ is the value of $m$ so that $\int_{-\infty}^{m} f(x)dx = 1/2$. The **expected value (or mean)** of $X$ is given by

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

The **variance** of $X$ is the expected value of $(X-\mu)^2$. The **standard deviation** is the square root of variance. In symbols,

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \text{ and } \sigma_X = \sqrt{\mathrm{Var}(X)}$$

Certain phenomenon can be modelled by exponentially decreasing probability distribution functions. $X$ is then said to follow an **exponential distribution**, and its probability density function is

$$f(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{1}{\mu} e^{-x/\mu} \text{ if } x \geq 0 \end{cases}$$

A **uniform distribution** models situations where each outcome is equally likely. In this case,

$$f(x) = \begin{cases} 0 \text{ if } x < a \text{ or } x > b \\ \frac{1}{b-a} \text{ if } a \leq x \leq b \end{cases}$$

Many situations involve the **normal distribution**. It is observed that many random variables approximately follow normal distributions. Then,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

Further assuming $f'$ and $g'$ to be continuous, and that the curve is traversed exactly once from $a$ to $b$, the **arc length** is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note that the natural parametrization shows that the formula in §6.2 is a special case of this formula. The arc length differential $ds = \sqrt{dx^2 + dy^2}$ also holds for parametric curves.

## 11.3 Polar coordinates

A polar coordinate system is an alternative to the Cartesian plane. We fix a point called the **origin (or pole)** and an initial ray. Then, for each point $P$ in the plane, we assign a **polar coordinate** pair $(r, \theta)$, where $r$ is the directed distance from the origin and $\theta$ is the angle measured counterclockwise from the initial ray.

Note that the polar coordinates of a point are not unique. If $(r, \theta)$ represents $P$, then so do $(r, \theta + 2n\pi)$ and $(-r, \theta + (2n+1)\pi)$, where $n \in \mathbb{Z}$.

Polar and Cartesian coordinates are related by the following formulae:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}$$

Some plane curves can be expressed both in Cartesian and polar equations. However, one of these forms may be simpler to work with.



## 11.4 Graphing polar coordinate equations

Symmetries can be useful in graph sketching. Let $(r, \theta)$ be a point on the graph. Then, if $(r, -\theta)$ is also on the graph, the graph is symmetric about the $x-$axis. If $(r, \pi - \theta)$ is also on the graph, the graph is symmetric about the $y-$axis. If $(-r, \theta)$ is also on the graph, the graph is symmetric about the origin.

The slope of a curve $r = f(\theta)$ at $(r, \theta)$, provided $dx/d\theta \neq 0$ there, is

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Substituting $f(\theta) = 0$, we see that the tangents at the origin have slope $\tan\theta_0$, where $f(\theta_0) = 0$. This tells us the shape of the graph near the origin.

An alternative approach to plotting a table of values is to first plot the graph of $r = f(\theta)$ in a $r\theta-$plane, then using the graph as a guide to sketch the graph in the $xy-$plane.