MULTIVARIATE FUNCTIONS

- Multivariate functions have more than one independent variable
- Most functions in economics are multivariate functions
- Function with two independent variables: $z = f(x, y)$
- Function with 3 independent variables: $z = f(x_1, x_2, x_3)$
- Extension to *n* independent variables: $z = f(x_1, x_2, \ldots, x_n)$
- Again consider the function $z = f(x, y)$
- This function needs to be graphed on 3 axes in 3D space and the resulting graph is a surface (e.g. Fig 14.2)
- If all three variables are raised to the power 1, and if there are no cross-products, then the resulting graph is called a plane (e.g. Fig 14.3)
- If one of the variables is kept constant, one can take ”slices” through the surface. These slices are called sections (e.g. Fig 14.1, Fig 14.4)
- These sections are known as iso-sections (if z is constant: iso-z section, if x constant, then iso-x section, if y constant, then iso-y section)
- Can represent these functions with lines/curves in 2D space

SOLUTION, EXAMPLE 1

1. $z = x^2 + 3xy + y^2 + 2$

- $\frac{\partial z}{\partial x} = 2x + 3y;\ \frac{\partial z}{\partial y} = 3x + 2y$
- $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 2;\ \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2$
- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 3;\ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 3$

2. $z = a \ln x + b \ln y$

- $\frac{\partial z}{\partial x} = \frac{a}{x};\ \frac{\partial z}{\partial y} = \frac{b}{y}$
- $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\mathrm{a}x^{-2};\ \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -by^{-2}$
- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0;\ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 0$

3. $z = x^{0.3}y^{0.7}$

- $\frac{\partial z}{\partial x} \Longrightarrow y^{0.7}$ is the multiplicative constant
- $\frac{\partial z}{\partial x} = y^{0.7} \times \frac{\partial}{\partial x}(x^{0.3}) \Longrightarrow \frac{\partial z}{\partial x} = y^{0.7} \times 0.3x^{-0.7}$
- $\therefore \frac{\partial z}{\partial x} = 0.3x^{-0.7}y^{0.7}$
- $z = x^{0.3}y^{0.7}$
- $\frac{\partial z}{\partial y} \Longrightarrow x^{0.3}$ is the multiplicative constant
- $\frac{\partial z}{\partial y} = x^{0.3} \times \frac{\partial}{\partial y}(y^{0.7}) \Longrightarrow \frac{\partial z}{\partial y} = x^{0.3} \times 0.7y^{-0.3}$
- $\therefore \frac{\partial z}{\partial y} = 0.7x^{0.3}y^{-0.3}$

DIMINISHING MU & ELASTICITY

- *Law of diminishing marginal utility*: as more of one good is consumed (while consumption of the other remains fixed), the extra utility from increased consumption becomes smaller and smaller
- Confirm: negative second partial derivatives
- Diminishing marginal utility of x: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) < 0$
- Diminishing marginal utility of y: $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) < 0$
- Elasticity of utility w.r.t. $x = e^x = \frac{MU_x}{AU_x}$, elasticity of utility w.r.t. $y = e^y = \frac{MU_y}{AU_y}$

FIRST-ORDER & SECOND-ORDER CONDITIONS

- Suppose that $z = f(x, y)$
- To find the minimum and/or maximum values of z:
- First-order (necessary) condition: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$
- Second-order (sufficient) condition:
- Maximum: $\frac{\partial^2 z}{\partial x^2} < 0, \frac{\partial^2 z}{\partial y^2} < 0, \frac{\partial^2 z}{\partial x \partial y} \times \frac{\partial^2 z}{\partial y \partial x} < \frac{\partial^2 z}{\partial x^2} \times \frac{\partial^2 z}{\partial y^2}$
- Minimum: $\frac{\partial^2 z}{\partial x^2} > 0, \frac{\partial^2 z}{\partial y^2} > 0, \frac{\partial^2 z}{\partial x \partial y} \times \frac{\partial^2 z}{\partial y \partial x} < \frac{\partial^2 z}{\partial x^2} \times \frac{\partial^2 z}{\partial y^2}$

SOLUTION, PROFIT MAXIMISATION

- $\frac{\partial^2 \pi}{\partial L^2} \times \frac{\partial^2 \pi}{\partial K^2} = 0.000004, \frac{\partial^2 \pi}{\partial L \partial K} \times \frac{\partial^2 \pi}{\partial K \partial L} = 0.0000038 < 0.000004$
- Therefore, profits are maximised if $K = 1024;\ L = 1024$, because FOC and SOC are met
- Note: for very small numbers, can use scientific notation
  - $0.48 \ = 4.8 \times 10^{-1}$
  - $0.0048 \ = 4.8 \times 10^{-3}$
  - $0.0000048 \ = 4.8 \times 10^{-6}$

PRICE DISCRIMINATION

- Then, obtain second partial derivatives and evaluate signs (must be negative for maximization), i.e. $\frac{\partial^2 \pi}{\partial q_1^2} < 0, \frac{\partial^2 \pi}{\partial q_2^2} < 0$
- Then obtain the cross-partial derivatives, and check if the product of these derivatives is less than the product of the second partial derivatives, i.e.: $\frac{\partial^2 \pi}{\partial q_1^2} \times \frac{\partial^2 \pi}{\partial q_2^2} > \frac{\partial^2 \pi}{\partial q_1 \partial q_2} \times \frac{\partial^2 \pi}{\partial q_2 \partial q_1}$
- Note that the FOC for profit maximization by a price discriminating monopolist can also be written as: $MR_1 = MC_1$, $MR_2 = MC_2$
- But $MC_1 = MC_2 = MC$
- This means that $MR_1 = MC$, $MR_2 = MC$
- Therefore $MR_1 = MR_2 = MC$ when price discriminating monopolist is maximizing profit

SOLUTION: PROFIT MAXIMISATION BY A 2-PRODUCT FIRM

- $R_a = 50q_a - q_a^2$ and $R_b = 95q_b - 3q_b^2$.
- $\pi = R_a + R_b - C = 50q_a - q_a^2 + 95q_b - 3q_b^2 - q_a^2 - 3q_aq_b - q_b^2$
- $\pi = 50q_a + 95q_b - 2q_a^2 - 4q_b^2 - 3q_aq_b$
- FOC: $\frac{\partial \pi}{\partial q_a} = \frac{\partial \pi}{\partial q_b} = 0$
- $\frac{\partial \pi}{\partial q_a} = 50 - 4q_a - 3q_b = 0$ [1]
- $\frac{\partial \pi}{\partial q_b} = 95 - 3q_a - 8q_b = 0$ [2]
- $3 \times [1]$ and $4 \times [2]$
- $150 - 12q_a - 9q_b = 0$ [1a]
- $380 - 12q_a - 32q_b = 0$ [2a]
- [1a] - [2a]: $150 - 12q_a - 9q_b - 380 + 12q_a + 32q_b = 0$
- $\Longrightarrow -230 = -23q_b, \therefore q_b = 10 \Longrightarrow q_a = 5$
- $p_a = 45, p_b = 65$