5.1.1 Similar Formulas

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{5.7}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{5.8}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{5.9}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} \tag{5.10}$$

$$\cosh\ z = \frac{e^{z} + e^{-z}}{2} \tag{5.11}$$

$$\sinh\ z = \frac{e^{z} - e^{-z}}{2} \tag{5.12}$$

$$\cosh iz = \cos z \tag{5.13}$$

$$\sinh iz = i \sin z \tag{5.14}$$

5.1.2 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta \tag{5.15}$$

From this it can be shown with $z = e^{i\theta}$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta \tag{5.16}$$

5.1.3 Roots of unity

$$z^n = 1 = e^{i2k\pi} \Rightarrow z = e^{i2k\pi/n} \tag{5.17}$$

where k is an integer and will take values $0, 1, 2, ..., n-1$ to give the n distinct roots

5.1.4 Logarithms

with $z = re^{i(\theta+2n\pi)}$

$$Ln\ z = ln\ r + i(\theta + 2n\pi) \tag{5.18}$$

Restricting to principal value by constraining the argument of z to lie between $-\pi to \pi$ we get singlevalued $Ln(z)$

The definition of complex number raised to a complex power is in terms of already defined complex functions:

$$t^z = e^{zln(t)} \tag{5.19}$$

5.2 Complex Functions

$$\lim_{z\to z_0} f(z) = \infty \text{ if and only if } \lim_{z\to z_0} \frac{1}{f(z)} = 0 \tag{5.20}$$

$$\lim_{z\to \infty} f(z) = w_0 \text{ if and only if } \lim_{z\to 0} f\frac{1}{z} = w_0 \tag{5.21}$$

$$\lim_{z\to \infty} f(z) = \infty \text{ if and only if } \lim_{z\to 0} \frac{1}{f(1/z)} = 0 \tag{5.22}$$

Since x and y are related to z and its complex conjugate z^* by

$$x = \frac{1}{2}(z + z^*) \qquad \text{and} \qquad y = \frac{1}{2i}(z - z^*), \tag{24.6}$$

we may formally regard any function $f = u + iv$ as a function of z and z^*, rather than x and y. If we do this and examine $\partial f/\partial z^*$ we obtain

$$\begin{aligned} \frac{\partial f}{\partial z^*} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \end{aligned} \tag{24.7}$$

Now, if f is analytic then the Cauchy–Riemann relations (24.5) must be satisfied, and these immediately give that $\partial f/\partial z^*$ is identically zero. Thus we conclude that if f is analytic then f cannot be a function of z^* and any expression representing an analytic function of z can contain x and y only in the combination $x + iy$, *not* in the combination $x - iy$.

5.2.1 Cauchy-Reimann Equations

Suppose that for a complex-valued function

$$f(z) = u(x, y) + iv(x, y) \tag{5.23}$$

$f'(z)$ exists at a point $z_0 = (x_0, y_0)$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Reimann Equations:

$$u_x = v_y \ , \ u_y = -v_x \tag{5.24}$$

And also

$$f'(z_0) = u_x + iv_x \tag{5.25}$$

- Differntiation of cauchy equations provides us with the fact that both u and v individually satisfy laplace equations in 2D

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{5.26}$$

- The family of 2D curves (in xy plane) $u(x, y) = constant$ and $v(x, y) = constant$ intersect at right angles to each other.
- For multivalued complex function $f(z)$, like $exp(z), Lnz, z^{1/2}$ etc. we can make them single valued and then use the analytic function analysis on it by concept of branch points and cuts
- Branch point is a point in Argand plane such that if z is varied in a closed loop enclosing the branch point, $f(z)$ doesn't return to its original value, although z does since $\theta \to \theta + 2\pi$ doesn't affect z.
- Branch cuts are lines in the argand diagram, finite or infinite that prevents us to ever make a closed loop enclosing a branch point.
- So, if we dont cross the branch cut, $f(z)$ remains single-valued.

5.2.2 Singularities

- Isolated singularity: when $f(z)$ isnt analytic at z_0 but analytic at all points in the neighbourhood.
- Pole: most imp isolated singularity
- If

$$f(z) = \frac{g(z)}{(z - z_0)^n} \tag{5.27}$$

such that $g(z_0) \neq 0$ and is analytic in neighbourhood of z_0 then its a pole of order n

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix} \tag{6.11}$$

Spherical

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{rsin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial f}{\partial z}\hat{\boldsymbol{\phi}} \tag{6.12}$$

$$\nabla.\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{rsin\phi}\frac{\partial A_2}{\partial \theta} + \frac{1}{rsin\phi}\frac{\partial}{\partial \phi}(sin\phi A_3) \tag{6.13}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 sin\phi}\begin{vmatrix} \hat{\mathbf{r}} & rsin\phi\hat{\boldsymbol{\theta}} & r\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & rsin\phi A_2 & rA_3 \end{vmatrix} \tag{6.14}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \phi^2}\right) \tag{6.15}$$

6.2.5 Fundamental theorems

The line integral of any gradient field doesn't depend on path but just on the endpoints. The little changes df add up (the integral on LHS) to give $f(b) - f(a)$

$$\int_a^b (\nabla f).d\mathbf{l} = f(b) - f(a) \tag{6.16}$$

The volume integral of a divergence over any given volumen is equal to the closed surface integral or the flux of the original vector field over the surface that bounds that volume.

$$\int_V (\nabla.\mathbf{A})d\tau = \oint_S \mathbf{A}.d\mathbf{a} \tag{6.17}$$

The surface integral of curl of a vector field is equal to the closed line integral of that field over the boundary of the surface. Since for a given boundary there can be infinite number of surfaces, it shows that surface integral of curls is only dependent on the boundary of the surface.

$$\int_S (\nabla \times \mathbf{A}).d\mathbf{a} = \oint \mathbf{A}.d\mathbf{l} \tag{6.18}$$

These operators are usually applied on scalar (gradient) and vector (divergence, curl) fields. Boldface notation is used for vector fields. f and g are scalar fields and k is just a scalar number

6.2.6 First derivatives

- Sum rules

$$\nabla(f+g) = \nabla f + \nabla g \tag{6.19}$$

$$\nabla.(\mathbf{A}+\mathbf{B}) = (\nabla.\mathbf{A}) + (\nabla.\mathbf{B}) \tag{6.20}$$

$$\nabla \times (\mathbf{A}+\mathbf{B}) = (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{B}) \tag{6.21}$$

- Multiplying by a constant rules

$$\nabla(kf) = k\nabla f \tag{6.22}$$

$$\nabla.(k\mathbf{A}) = k(\nabla.\mathbf{A}) \tag{6.23}$$

$$\nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}) \tag{6.24}$$

Chapter 7

Coordinate Systems

7.1 Cartesian coordinates

Straightforward and easy to understand formulas for all things.

- Point P(x,y,z)

  where x, y and z denotes the distances along the x,y and z axes respectively.
  The position vector of any point P is given by OP $= x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ where the unit vectors are $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ that are fixed in magnitude($= 1$) and direction also. So in any problem with changing position/displacement vector, their time derivative must be zero

- Distance Formula

  Between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{7.1}$$

- Equation of plane with unit normal vector $\hat{n}$ and a given point on plane with position vector $\mathbf{r}'$:

$$(\mathbf{r} - \mathbf{r}').\hat{n} = 0 \tag{7.2}$$

- In component form:

$$px + qy + sz = d \quad \hat{n} = (p, q, s) \quad \mathbf{r}.\hat{n} = d \tag{7.3}$$

  where $d =$ perpendicular distance of plane from origin

- In intercept form:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{7.4}$$

- Relation b/w unit normal and intercepts:

$$\hat{n} \propto \frac{\hat{i}}{a} + \frac{\hat{j}}{b} + \frac{\hat{k}}{c} \tag{7.5}$$

- Angle between two planes: $l_1x + m_1y + n_1z = d_1$ and $l_2x + m_2y + n_2z = d_2$ is:

$$\cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}} \tag{7.6}$$

- Laplacian

$$\nabla.(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{7.7}$$

- Laplacian of a vector

$$\nabla^2\mathbf{v} \equiv (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z} \tag{7.8}$$

3D spherical polar

Line element:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{7.21}$$

Volume element:

$$dV = r^2 \sin\theta d\phi d\theta dr \tag{7.22}$$

Cylindrical

Line element:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{7.23}$$

Volume element:

$$dV = \rho d\phi d\rho dz \tag{7.24}$$

7.4 Coordinate transformations and unit vectors

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

12.2 Fourier integral and transforms

Fourier cosine integral which applies to an even function $f(x)$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos\ wx\ dx \quad f(x) = \int_0^\infty A(w)\cos\ wx\ dw \tag{12.6}$$

Fourier sine integral which applies to an odd function $f(x)$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin\ wx\ dx \quad f(x) = \int_0^\infty B(w)\sin\ wx\ dw \tag{12.7}$$

For a general function $f(x)$, we have representation in terms of its fourier integral. Here the series of eq. 11.1 transforms to an integral as we take $L \to \infty$

$$f(x) = \int_0^\infty [A(w)\cos\ wx + B(w)\sin\ wx]dw \tag{12.8}$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos\ wv\ dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\ wv\ dv \tag{12.9}$$

Writing in complex and doing few tricks we get:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-ikx}dx \tag{12.10}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(k)e^{ikx}dk \tag{12.11}$$

12.2.1 Fourier sine and cosine transforms

$$\hat{f}_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\ dx \tag{12.12}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(k)\sin kx\ dk \tag{12.13}$$

$$\hat{f}_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\ dx \tag{12.14}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(k)\cos kx\ dk \tag{12.15}$$

Properties

$$\mathcal{F}_s\{f'(x)\} = -k\mathcal{F}_c\{f(x)\} \tag{12.16}$$

$$\mathcal{F}_c\{f'(x)\} = k\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0) \tag{12.17}$$

$$\mathcal{F}_c\{f''(x)\} = -k^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0) \tag{12.18}$$

$$\mathcal{F}_s\{f''(x)\} = -k^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}kf(0) \tag{12.19}$$

$$\frac{d}{dk}\mathcal{F}_s(k) = \mathcal{F}_c\{xf(x)\} \tag{12.20}$$

$$\frac{d}{dk}\mathcal{F}_c(k) = -\mathcal{F}_s\{xf(x)\} \tag{12.21}$$

13.1.3 Poisson distribution

This distribution is actually a discrete distribution cause the random variable is discrete. But that random variable is spread over a continuum. Like the number of calls that we will receive over some particular time interval.

$$P_x(t) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \tag{13.22}$$

This is the probability of receiving x number of calls in a time period of t given that λ is the average number of calls per unit time.
Mean and variance are both λ

13.1.4 Gaussian distribution

Many random variables occuring in physical sciences and others follow this distribution exactly or approximately.
The probability density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{13.23}$$

Mean and variance can be changed by shift of origin and scaling. Then with standard variable $Z = (X-\mu)/\sigma$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} exp\left(-z^2/2\right) \tag{13.24}$$

The cumulative probability function cant be solved analytically but is tabulated as $\Phi(z)$ for different z values.

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{13.25}$$

$$Pr(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \tag{13.26}$$

13.2 Random walk

- Simplest random walk: Walk unit length steps by flipping a coin. If head comes move forward, tail comes move backwards.
- Parameters: N = total number of steps taken. l = length of each step. n = the integer (if $l = 1$) where you landed at the end.
- Pattern on analysis probabilities of different possible ns when N changes.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f_0(n)$						1					
$2f_1(n)$					1		1				
$2^2f_2(n)$				1		2		1			
$2^3f_3(n)$			1		3		3		1		
$2^4f_4(n)$		1		4		6		4		1	
$2^5f_5(n)$	1		5		10		10		5		1

- Thus, the coefficients are binomial
- If number of heads = $n_{forward}$ and tails = $n_{backward}$ get fixed then final position is fixed $n = n_{forward} - n_{backward}$.