$$\Rightarrow Z_k = 2\left[\cos\left(2\pi k + \frac{\pi}{2}\right) + i \sin\left(2\pi k + \frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

where $k = 0, 1, 2.$

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$$\therefore Z_k = 2\left[\cos\left(\frac{4k\pi + \pi}{6}\right) + i \sin\left(\frac{4k\pi + \pi}{6}\right)\right]$$

So put $k = 0, 1, 2$, then required three roots are given by.

for $k=0$, $Z_0 = 2\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right] = 2\left\{\frac{\sqrt{3}}{2} + \frac{i \cdot 1}{2}\right\} \Rightarrow \boxed{\sqrt{3} + i = Z_0}$

for $k=1$, $Z_1 = 2\left[\cos\left(\frac{4\pi+\pi}{6}\right) + i \sin\left(\frac{4\pi+\pi}{6}\right)\right] = 2\left\{\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right]$

$$= 2\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \Rightarrow \boxed{Z_1 = -\sqrt{3} + i}$$

and 3rd root is obtained by $k = 2$, we get

$$Z_2 = 2\left\{\cos\left(\frac{8\pi+\pi}{6}\right) + i \sin\left(\frac{8\pi+\pi}{6}\right)\right] = 2\left[\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right]$$

$$= 2\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right] = 2(0 - i) \Rightarrow \boxed{Z_2 = -2i}$$

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Part-(ii) Find fourth roots of each of the following complex number.

(a) $-16i$, (b) 64, (c) $-2\sqrt{3} + 2i$

(a) Since we have find four fourth roots of $-16i$

So put $z^4 = -16i = 16(0 - i)$

$$= 2^4\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right]$$

$x = 0, y = -1$ $r = \sqrt{0^2 + (-1)^2} = 1$

$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1} 0$

$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \Rightarrow \theta = \cos^{-1}(-1)$ $\Rightarrow \theta = -\frac{\pi}{2}$

(Since lies in IV th Q ∴ $-\theta$ Hence $-\frac{\pi}{2}$)

$$z^4 = 2^4\left[\cos\left(2\pi k - \frac{\pi}{2}\right) + i \sin\left(2k\pi - \frac{\pi}{2}\right)\right]$$

So fourth root of $-16i$ is

$$Z_k = 2\left[\cos\left(2\pi k - \frac{\pi}{2}\right) + i \sin\left(2k\pi - \frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

where $k = 0, 1, 2, 3$

$$= 2\left[\cos\frac{1}{4}\left(\frac{4k\pi - \pi}{2}\right) + i \sin\left(\frac{4k\pi - \pi}{2}\right)\frac{1}{4}\right]$$

$$Z_k = 2\left[\cos\left(\frac{4k\pi - \pi}{8}\right) + i \sin\left(\frac{4k\pi - \pi}{8}\right)\right], \quad k = 0, 1, 2, 3 \rightarrow (1)$$

Easy

2nd Method: Now $x^4+x^2+1=0$

$x^4+x^2+1=0$

$x^4+x^2+x^2-x^2+1=0$

$x^4+2x^2+1-x^2=0$

$(x^2+1)^2-x^2=0$

$(x^2+1+x)(x^2+1-x)=0$

Either $x^2+1+x=0$

$x=\frac{-1\pm\sqrt{1-4}}{2}$

$x=\frac{-1\pm i\sqrt{3}}{2}$

OR $x^2+1-x=0$

$x=\frac{1\pm\sqrt{1-4}}{2}$

$=\frac{1\pm i\sqrt{3}}{2}$

Put $x^2=y$

$y^2+y+1=0$

$y=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

x both by 2, by completing square

$x^2=\frac{-2\pm2\sqrt{-3}}{4}$

$=\frac{-2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$=\frac{(1-3)\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\frac{1^2+(\sqrt{-3})^2\pm2\cdot1\cdot\sqrt{-3}}{4}$

$x^2=\left(\frac{1\pm\sqrt{-3}}{2}\right)^2$

$x=\pm\left(\frac{1\pm\sqrt{-3}}{2}\right)$

$\Rightarrow x=\frac{1\pm\sqrt{3}i}{2},\ \frac{-1\pm\sqrt{3}i}{2}$

$x=\frac{1+\sqrt{3}i}{2},\ \frac{1-\sqrt{3}i}{2},\ \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2}$

We see that of twelve roots of $x^{12}=-1$

x_2, x_4, x_8 and x_{10} satisfy the roots of equation $x^4+x^2+1=0$

x——————x

Q. 10 Expand the following in series of sines or cosines of multiples of θ

$(a+b)^n=a^n+na^{n-1}\cdot b+\frac{n(n-1)}{\underline{2}}a^{n-2}b^2+\cdots\cdots b^n$

(i) $\cos^4\theta$

Sol: let $x=\cos\theta+i\sin\theta$, then $\frac{1}{x}=\frac{1}{\cos\theta+i\sin\theta}$

$=(\cos\theta+i\sin\theta)^{-1}$

$\frac{1}{x}=\cos\theta-i\sin\theta$

$\Rightarrow x+\frac{1}{x}=2\cos\theta$

So $(2\cos\theta)^4=\left(x+\frac{1}{x}\right)^4$

$2^4\cos^4\theta=x^4+4x^3\left(\frac{1}{x}\right)+\frac{4\cdot3}{2\cdot1}\cancel{x^2}\cdot\frac{1}{\cancel{x^2}}+\frac{4\cdot3\cdot2}{3\cdot2\cdot1}x\cdot\frac{1}{x^3}+\frac{1}{x^4}$

$=x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

(V) Q-10 $\sin^9\theta = ?$

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Sol:- let $x = \cos\theta + i\sin\theta$, then

$\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow x - \frac{1}{x} = 2i\sin\theta$

$\Rightarrow (2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$

or $2^9 i^9 \sin^9\theta = x^9 - 9x^8\frac{1}{x} + \frac{9\cdot 8\, x^7}{2\cdot 1\cdot x^2} - \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\frac{x^6}{x^3} + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\frac{x^5}{x^4}$

$- \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^4}{x^5} + \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^3}{x^6} - \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\frac{x^2}{x^7}$

$+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{8!}\frac{x}{x^8} - \frac{1}{x^9}$

$2^9 i^{8+1}\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$2^9 i\sin^9\theta = x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x}$

$+ \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$

$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right)$

$+ 126\left(x - \frac{1}{x}\right)$

$= 2i\sin 9\theta - 9(2i\sin 7\theta) + 36(2i\sin 5\theta) - 84(2i\sin 3\theta)$

$+ 126(2i\sin\theta)$

$2^9 i\sin^9\theta = 2i\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$

$$\boxed{\sin^9\theta = \frac{1}{2^8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]}$$

Ans.

(vi) Q-10 $\sin^6\theta\cos^2\theta = ?$

Sol:- let $x = \cos\theta + i\sin\theta$ and then $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Big\}$ $\Rightarrow x + \frac{1}{x} = 2\cos\theta$

and $x - \frac{1}{x} = 2i\sin\theta$

$$2^4\left[\cos^4\theta+\sin^4\theta\right] = 2\left[x^4+6+\frac{1}{x^4}\right] \qquad (2^4 = 1)$$

$$\cos^4\theta+\sin^4\theta = \frac{1}{2^3}\left[\left(x^4+\frac{1}{x^4}\right)+6\right] = \frac{1}{8}\left[2\cos 4\theta+6\right]$$

$$\boxed{\cos^4\theta+\sin^4\theta = \frac{1}{4}\left[\cos 4\theta+3\right]}$$

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— x — Ans — x —

Q.12 Prove that $64(\cos^8\theta+\sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$

Sol: Let $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$ $\Rightarrow$ $x+\frac{1}{x} = 2\cos\theta$ and $x-\frac{1}{x} = 2i\sin\theta$

So $(2\cos\theta)^8+(2i\sin\theta)^8 = \left[\left(x+\frac{1}{x}\right)^8+\left(x-\frac{1}{x}\right)^8\right]$

$$2^8\left[\cos^8\theta+\sin^8\theta\right] = \left\{\begin{array}{l} x^8+8x^7\cdot\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}+\frac{8\cdot7\cdot6}{3\cdot2}x^5\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ +\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}x^3\cdot\frac{1}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\cdot\frac{1}{x^7}+\frac{1}{x^8} \\ x^8-8x^7\cdot\frac{1}{x}+\frac{8\cdot7}{2}x^6\cdot\frac{1}{x^2}-\frac{8\cdot7\cdot6}{3\cdot2}x^5\cdot\frac{1}{x^3}+\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}x^4\cdot\frac{1}{x^4} \\ -\frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2}\frac{x^3}{x^5}+\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6\cdot5\cdot4\cdot3\cdot2}x^2\cdot\frac{1}{x^6}-\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x\cdot\frac{1}{x^7}+\frac{1}{x^8} \end{array}\right\}$$

$$2^8\left[\cos^8\theta+\sin^8\theta\right] = 2\left[x^8+28x^6\cdot\frac{1}{x^2}+70+28\cdot x^2\cdot\frac{1}{x^6}+\frac{1}{x^8}\right]$$

$$= 2\left[\left(x^8+\frac{1}{x^8}\right)+28\left(x^4+\frac{1}{x^4}\right)+70\right]$$

$$= 2\left[2\cos 8\theta+28(2\cos 4\theta)+70\right]$$

$$2^6(\cos^8\theta+\sin^8\theta) = \cancel{2}\left[\cos 8\theta+28\cos 4\theta+35\right]$$

$$\boxed{\Rightarrow\ 64(\cos^8\theta+\sin^8\theta) = \cos 8\theta+28\cos 4\theta+35}$$

Proved.

Q.17 Prove the following relations ($\forall$ m, n belongs to $\mathbb{Z}$) mean m & n are integers

(i) $z^m z^n = z^{m+n}$ $\boxed{1.2\text{-}35}$

Proof let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$

$z^m = r^m(\cos m\theta + i\sin m\theta)$

Similarly $z^n = r^n(\cos n\theta + i\sin n\theta)$

Then $L.H.S. = z^m \cdot z^n$

$= r^m[\cos m\theta + i\sin m\theta]\, r^n[\cos n\theta + i\sin n\theta]$

$= r^m r^n[\cos m\theta + i\sin m\theta][\cos n\theta + i\sin n\theta]$

$= r^{m+n}\left[(\cos m\theta\cos n\theta - \sin m\theta\sin n\theta) + i(\sin m\theta\cos n\theta + \sin n\theta\cos m\theta)\right]$

$= r^{m+n}[\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)]$

$= r^{m+n}[\cos(m+n)\theta + i\sin(m+n)\theta]$

$= r^{m+n}(\cos\theta + i\sin\theta)^{m+n}$ by De-moivre's Th.

$= z^{m+n} = R.H.S.$

(ii) $(z^m)^n = z^{mn}$

Proof:- let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^m = r^m(\cos\theta + i\sin\theta)^m$ (De-moivre's Th.)

$z^m = r^m(\cos m\theta + i\sin m\theta)$

$\Rightarrow (z^m)^n = r^{mn}(\cos m\theta + i\sin m\theta)^n$ (De-moivre's Th.)

$$L.H.S = (Z^m)^n = r^{mn}\left[\cos mn\theta + i \sin mn\theta\right] = r^{mn}\left[\cos\theta + i\sin\theta\right]^{mn}$$

$$= Z^{mn} = R.H.S.$$

×————×

(iii) $(Z_1 Z_2)^n = Z_1^n Z_2^n$

PROOF let $Z_1 = r_1[\cos\theta_1 + i\sin\theta_1]$ and $Z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$

Hence $Z_1 \cdot Z_2 = r_1[\cos\theta_1 + i\sin\theta_1] \cdot r_2[\cos\theta_2 + i\sin\theta_2]$

$$= r_1 r_2 [\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$$

$$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$Z_1 Z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$L.H.S = (Z_1 Z_2)^n = r_1^n r_2^n [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]^n$$

by Demoivre's

$\because Z = r(\cos\theta + i\sin\theta)$
$\Rightarrow Z^n = [r(\cos\theta + i\sin\theta)]^n$
$= r^n(\cos\theta + i\sin\theta)^n$
$= r^n[\cos n\theta + i\sin n\theta]$

$$= r_1^n r_2^n [\cos n(\theta_1+\theta_2) + i\sin n(\theta_1+\theta_2)]$$

$$= r_1^n r_2^n [\cos(n\theta_1 + n\theta_2) + i\sin(n\theta_1 + n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1 \cos n\theta_2 - \sin n\theta_1 \sin n\theta_2) + i(\sin n\theta_1 \cos n\theta_2 + \cos n\theta_1 \sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1 \cos n\theta_2 + i^2 \sin n\theta_1 \sin n\theta_2) + (i\sin n\theta_1 \cos n\theta_2 + i\cos n\theta_1 \sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1 \cos n\theta_2 + i\cos n\theta_1 \sin n\theta_2) + (i\sin n\theta_1 \cos n\theta_2 + i^2 \sin n\theta_1 \sin n\theta_2)]$$

$$= r_1^n r_2^n [\cos n\theta_1(\cos n\theta_2 + i\sin n\theta_2) + i\sin n\theta_1(\cos n\theta_2 + i\sin n\theta_2)]$$

$$= r_1^n r_2^n [(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 + i\sin n\theta_2)]$$

$$= r_1^n(\cos n\theta_1 + i\sin n\theta_1) \cdot r_2^n(\cos n\theta_2 + i\sin n\theta_2)$$

$$= r_1^n(\cos\theta_1 + i\sin\theta_1)^n \cdot r_2^n(\cos\theta_2 + i\sin\theta_2)^n$$

$$= Z_1^n \, Z_2^n = R.H.S$$