(v) $\Rightarrow$(iv) Assume that $\|F(x)\| \leq \alpha\|x\|$, $\forall x \in X$. Given $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{\alpha}$, so that

$$x, y \in X \text{ with } \|x-y\| < \delta \Rightarrow \|F(x) - F(y)\| = \|F(x-y)\| \leq \alpha\|x-y\| < \alpha\,\delta = \alpha\,\frac{\epsilon}{\alpha} = \epsilon.$$

Therefore, F is uniformly continuous on X.

(iv) $\Rightarrow$(iii) Clearly, F is uniformly continuous on $X \Rightarrow F$ is continuous on X.

(iii) $\Rightarrow$(ii) Clearly, F is continuous on $X \Rightarrow F$ is continuous at 0.

(ii) $\Rightarrow$(i) Assume that F is continuous at 0. For $\epsilon = 1$, there exists $\delta > 0$ such that $F(U(0,\delta)) \subset U(0,1)$. If we take $r = \frac{\delta}{2}$, then $\overline{U}(0,r) \subset U(0,\delta)$ and hence F is bounded on $\overline{U}(0,r)$.

(iii) $\Rightarrow$(vi) Assume that F is continuous on X. Since $Z(F) = F^{-1}(\{0\})$ and $\{0\}$ is closed in Y, we get that $Z(F)$ is closed in X. Using (v), there exists $\alpha > 0$ such that $\|F(x)\| \leq \alpha\|x\|$, $\forall x \in X$. For arbitrary $x \in X$ and arbitrary $z \in Z(F)$, we have

$$\|\tilde{F}(x + Z(F))\| = \|\tilde{F}(x + z + Z(F))\| = \|F(x+z)\| \leq \alpha\|x+z\|.$$

Since $z \in Z(F)$ is arbitrary, we have

$$\frac{1}{\alpha}\|\tilde{F}(x + Z(F))\| \leq \inf\{\|x+z\| : z \in Z(F)\} = |||x + Z(F)|||.$$

Therefore, using (v), we get that $\tilde{F}$ is continuous.

(vi) $\Rightarrow$(iii) Assume that $Z(F)$ is closed and the linear map $\tilde{F} : X/Z(F) \to Y$ defined by $\tilde{F}(x + Z(F)) = F(x)$, $\forall x \in X$ is continuous. Then, using (v), there exists $\alpha > 0$ such that

$$\|\tilde{F}(x + Z(F))\| \leq \alpha\ |||x + Z(F)|||,\ \forall x \in X.$$

Therefore, using $|||x + Z(F)||| \leq \|x\|$, we get

$$\|F(x)\| = \|\tilde{F}(x + Z(F))\| \leq \alpha\ |||x + Z(F)||| \leq \alpha\,\|x\|,\ \forall x \in X.$$

Hence the theorem follows.

Definition. A function $f : X \to Y$ is a homeomorphism if f is one-to-one, f is continuous, and $f^{-1} : f(X) \to X$ is continuous.

Corollary. Let $F : X \to Y$ be a linear transformation.

(a) (i) F is a homeomorphism iff there exist $\alpha > 0$ and $\beta > 0$ such that $\beta\,\|x\| \leq \|F(x)\| \leq \alpha\,\|x\|$, $\forall x \in X$.

(ii) If F is a homeomorphism from X onto Y, then X is complete iff Y is complete.

(b) Let X and Y be normed spaces with dimension of X is n, for some $n \in \mathbb{N}$.

(i) Every bijective linear map from X to Y is a homeomorphism.

(ii) All norms on X are equivalent.