3.4. Domain and "biggest possible domain." In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

4.3. Example: the equation alone does not determine the function. Define $y = h(x)$ to be the solution of

$$x^2 + y^2 = 1.$$

If $x > 1$ or $x < -1$ then $x^2 > 1$ and there is no solution, so $h(x)$ is at most defined when $-1 \le x \le 1$. But when $-1 < x < 1$ there is another problem: not only does the equation have a solution, but it even has two solutions:

$$x^2 + y^2 = 1 \iff y = \sqrt{1 - x^2} \text{ or } y = -\sqrt{1 - x^2}.$$

The rule which defines a function must be unambiguous, and since we have not specified which of these two solutions is $h(x)$ the function is not defined for $-1 < x < 1$.

One can fix this by making a choice, but there are many possible choices. Here are three possibilities:

$$h_1(x) = \text{the nonnegative solution } y \text{ of } x^2 + y^2 = 1$$
$$h_2(x) = \text{the nonpositive solution } y \text{ of } x^2 + y^2 = 1$$
$$h_3(x) = \begin{cases} h_1(x) & \text{when } x < 0 \\ h_2(x) & \text{when } x \ge 0 \end{cases}$$

4.4. Why use implicit functions? In all the examples we have done so far we could replace the implicit description of the function with an explicit formula. This is not always possible or if it is possible the implicit description is much simpler than the explicit formula. For instance, you can define a function f by saying that $y = f(x)$ if and only if

(1) $$y^3 + 3y + 2x = 0.$$

This means that the recipe for computing $f(x)$ for any given x is "solve the equation $y^3 + 3y + 2x = 0$." E.g. to compute $f(0)$ you set $x = 0$ and solve $y^3 + 3y = 0$. The only solution is $y = 0$, so $f(0) = 0$. To compute $f(1)$ you have to solve $y^3 + 3y + 2 \cdot 1 = 0$, and if you're lucky you see that $y = -1$ is the solution, and $f(1) = -1$.

In general, no matter what x is, the equation (1) turns out to have exactly one solution y (which depends on x, this is how you get the function f). Solving (1) is not easy. In the early 1500s Cardano and Tartaglia discovered a formula[1] for the solution. Here it is:

$$y = f(x) = \sqrt[3]{-x + \sqrt{1 + x^2}} - \sqrt[3]{x + \sqrt{1 + x^2}}.$$

The implicit description looks a lot simpler, and when we try to differentiate this function later on, it will be much easier to use "implicit differentiation" than to use the Cardano-Tartaglia formula directly.

4.5. Inverse functions. If you have a function f, then you can try to define a new function f^{-1}, the so-called ***inverse function of f***, by the following prescription:

(2) For any given x we say that $y = f^{-1}(x)$ if y is the solution to the equation $f(y) = x$.

So to find $y = f^{-1}(x)$ you solve the equation $x = f(y)$. If this is to define a function then the prescription (2) must be unambiguous and the equation $f(y) = x$ has to have a solution and cannot have more than one solution.

Find a formula for f. What are the domain and range of f?

12. Find a formula for the function f which is defined by

$$y = f(x) \iff 2x + 2xy + y^2 = 5 \text{ and } y > -x.$$

Find the domain of f.

13. Use a calculator to compute $f(1.2)$ in three decimals where f is the implicitly defined function from §4.4. (There are (at least) two different ways of finding $f(1.2)$)

14. Group Problem.

(a) True or false:
for all x one has $\sin \arcsin x = x$?

(b) True or false:
for all x one has $\arcsin \sin x = x$?

15. On a graphing calculator plot the graphs of the following functions, and explain the results. (Hint: first do the previous exercise.)

$$f(x) = \arcsin(\sin x), \quad -2\pi \le x \le 2\pi \; g(x)$$
$$= \arcsin(x) + \arccos(x), \qquad 0 \le x \le 1$$
$$h(x) = \arctan \frac{\sin x}{\cos x}, \quad |x| < \pi/2$$
$$k(x) = \arctan \frac{\cos x}{\sin x}, \quad |x| < \pi/2$$
$$l(x) = \arcsin(\cos x), \quad -\pi \le x \le \pi$$
$$m(x) = \cos(\arcsin x), \quad -1 \le x \le 1$$

16. Find the inverse of the function f which is given by $f(x) = \sin x$ and **whose domain is** $\pi \le x \le 2\pi$. Sketch the graphs of both f and f^{-1}.

17. Find a number a such that the function $f(x) = \sin(x + \pi/4)$ with domain $a \le x \le a + \pi$ has an inverse. Give a formula for $f^{-1}(x)$ using the arcsine function.

18. Draw the graph of the function h_3 from §4.3.

19. A function f is given which satisfies

$$f(2x + 3) = x^2$$

for all real numbers x.

Compute

(a) $f(0)$ (b) $f(3)$ (c) $f(x)$

(d) $f(y)$ (e) $f(f(2))$

where x and y are arbitrary real numbers. What are the range and domain of f?

20. A function f is given which satisfies

$$f\left(\frac{1}{x+1}\right) = 2x - 12.$$

for all real numbers x.

Compute

(a) $f(1)$ (b) $f(0)$ (c) $f(x)$

(d) $f(t)$ (e) $f(f(2))$

where x and t are arbitrary real numbers.

What are the range and domain of f?

21. Does there exist a function f which satisfies

$$f(x^2) = x + 1$$

for **all** real numbers x?

* * *

The following exercises review precalculus material involving quadratic expressions $ax^2 + bx + c$ in one way or another.

22. Explain how you "complete the square" in a quadratic expression like $ax^2 + bx$.

23. Find the range of the following functions:

$$f(x) = 2x^2 + 3$$
$$g(x) = -2x^2 + 4x$$
$$h(x) = 4x + x^2$$
$$k(x) = 4\sin x + \sin^2 x$$
$$l(x) = 1/(1 + x^2)$$
$$m(x) = 1/(3 + 2x + x^2).$$

24. Group Problem.

For each real number a we define a line l_a with equation $y = ax + a^2$.

(a) Draw the lines corresponding to $a = \ldots$