

By the chain rule we therefore get

$$\begin{aligned} \frac{da^x}{dx} &= \frac{d2^{x \cdot \log_2 a}}{dx} \\ &= C 2^{x \cdot \log_2 a} \frac{dx \cdot \log_2 a}{dx} \\ &= (C \log_2 a) 2^{x \cdot \log_2 a} \\ &= (C \log_2 a) a^x. \end{aligned}$$

So the derivative of  $a^x$  is just some constant times  $a^x$ , the constant being  $C \log_2 a$ . This is essentially our formula for the derivative of  $a^x$ , but one can make the formula look nicer by introducing a special number, namely, we define

$$e = 2^{1/C} \text{ where } C = \lim_{\Delta x \rightarrow 0} \frac{2^{\Delta x} - 1}{\Delta x}.$$

One has

$$e \approx 2.718\ 281\ 818\ 459 \dots$$

This number is special because if you set  $a = e$ , then

$$C \log_2 a = C \log_2 e = C \log_2 2^{1/C} = C \cdot \frac{1}{C} = 1,$$

and therefore the derivative of the function  $y = e^x$  is

$$(41) \quad \frac{de^x}{dx} = e^x.$$

Read that again: the function  $e^x$  is its own derivative!

The logarithm with base  $e$  is called the **Natural Logarithm**, and is written

$$\ln x = \log_e x.$$

Thus we have

$$(42) \quad \frac{e^{\ln x} = x}{e^{\ln x} = x} \quad \frac{\ln e^x = x}{\ln e^x = x}$$

where the second formula holds for all real numbers  $x$  but the first one only makes sense for  $x > 0$ .

For any positive number  $a$  we have  $a = e^{\ln a}$ , and also

$$a^x = e^{x \ln a}.$$

By the chain rule you then get

$$(43) \quad \frac{da^x}{dx} = a^x \ln a.$$

## 6. Derivatives of Logarithms

Since the natural logarithm is the inverse function of  $f(x) = e^x$  we can find its derivative by implicit differentiation. Here is the computation (which you should do yourself)

The function  $f(x) = \log_a x$  satisfies

$$a^{f(x)} = x$$

Differentiate both sides, and use the chain rule on the left,

$$(\ln a)a^{f(x)}f'(x) = 1.$$

Then solve for  $f'(x)$  to get

$$f'(x) = \frac{1}{(\ln a)a^{f(x)}}.$$

Finally we remember that  $a^{f(x)} = x$  which gives us the derivative of  $a^x$

$$\frac{da^x}{dx} = \frac{1}{x \ln a}.$$