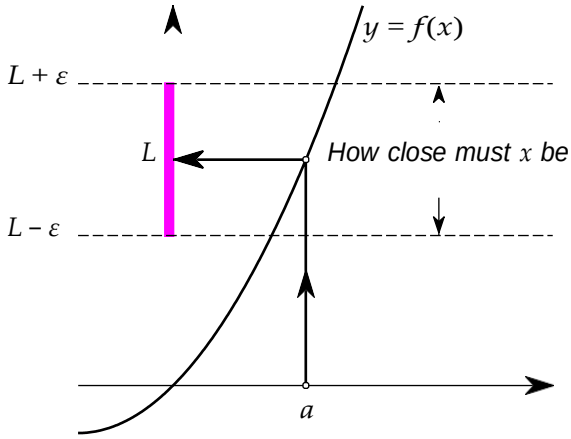
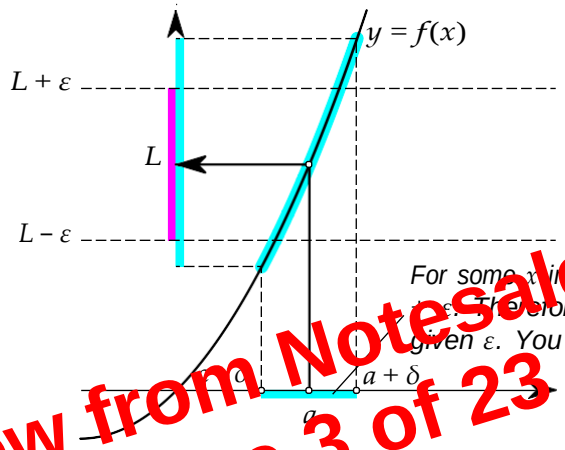
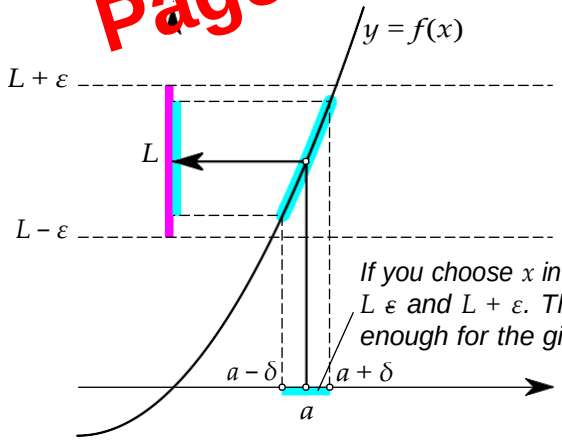


How close must x be to a for $f(x)$ to end up in this range?



For some x in this interval $f(x)$ is not between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is too big for the given ε. You need a smaller δ.



If you choose x in this interval then $f(x)$ will be between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is small enough for the given ε.

2.3. The limit $\lim_{x\to 1} x^2 = 1$ and the "don't choose $\delta > 1$" trick. We have $f(x) = x^2$, $a = 1$, $L = 1$, and again the question is, "how small should $|x - 1|$ be to guarantee $|x^2 - 1| < \varepsilon$?"

We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x + 1| \cdot |x - 1|.$$

4.1. Left and right limits. When we let "x approach a" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of $f(x)$ as x approaches a through values larger than a, then we write

$$\lim_{x \& a} f(x) \text{ or } \lim_{x \to a+} f(x) \text{ or } \lim_{x \to a+0} f(x) \text{ or } \lim_{x \to a, x > a} f(x).$$

All four notations are in use. Similarly, to designate the value which $f(x)$ approaches as x approaches a through values below a one writes

$$\lim_{x \% a} f(x) \text{ or } \lim_{x \to a-} f(x) \text{ or } \lim_{x \to a-0} f(x) \text{ or } \lim_{x \to a, x < a} f(x).$$

The precise definition of right limits goes like this:

4.2. Definition of right-limits. *Let f be a function. Then*

$$\lim_{x \& a} f(x) = L. \tag{8}$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

holds for all x in the domain of f.

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way. The following theorem tells you how to use one-sided limits to decide if a function $f(x)$ has a limit at $x = a$.

4.3. Theorem. *If both one-sided limits*

$$\lim_{x \& a} f(x) = L_+, \text{ and } \lim_{x \% a} f(x) = L_-$$

exist, then

$$\lim_{x \to a} f(x) \text{ exists} \iff L_+ = L_-$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

4.4. Limits at infinity. Instead of letting x approach some finite number, one can let x become "larger and larger" and ask what happens to $f(x)$. If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x \to \infty} f(x) = L, \quad \text{or} \quad \lim_{x \uparrow \infty} f(x) = L, \quad \text{or} \quad \lim_{x \% \infty} f(x) = L.$$

("The limit for x going to infinity is L.")

4.5. Example – Limit of $1/x$. The larger you choose x, the smaller its reciprocal $1/x$ becomes. Therefore, it seems reasonable to say

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Here is the precise definition:

4.6. Definition of limit at ∞. *Let f be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\varepsilon > 0$ one can find an A such that*

$$x > A \implies |f(x) - L| < \varepsilon$$

for all x, then we say that the limit of $f(x)$ for $x \to \infty$ is L.

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a, we now have a number A which says how large x should be, which is a way of saying "how close x should be to infinity."

So we see that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$$\lim_{x\to\infty} \; x + 2 - \frac{1}{x} \;=\; \lim_{x\to\infty} \frac{x + 2 - \frac{1}{x}}{1 + 2/x} \cdot (1 + 2/x)$$

would also have to have a limit.

8. What's in a name?

There is a big difference between the variables x and a in the formula

$$\lim_{x\to a} 2x + 1,$$

namely a is a ***free variable***, while x is a ***dummy variable*** (or "placeholder" or a "bound variable.")

The difference between these two kinds of variables is this:

- if you replace a dummy variable in some formula consistently by some other variable then the value of the formula does not change. On the other hand, it never makes sense to substitute a number for a dummy variable.
- the value of the formula may depend on the value of the free variable.

To understand what this means consider the example $\lim_{x\to a} 2x + 1$ again. The limit is easy to compute:

$$\lim_{x\to a} 2x + 1 = 2a + 1.$$

If we replace x by, say u (systematically) then we get

$$\lim_{u\to a} 2u + 1$$

which is again equal to $2a + 1$. This computation says that *if some number gets close to a then two times that number plus one gets close to $2a + 1$.* This is a very wordy way of expressing the formula, and you can shorten things by giving a name (like x or u) to the number which approaches a. But the result of our computation shouldn't depend on the name we choose, i.e. it doesn't matter if we call it x or u.

Since the name of the variable x doesn't matter it is called a dummy variable. Some prefer to call x a bound variable, meaning that in

$$\lim_{x\to a} 2x + 1$$

the x in the expression $2x + 1$ is bound to the x written underneath the limit – you can't change one without changing the other.

Substituting a number for a dummy variable usually leads to complete nonsense. For instance, let's try setting $x = 3$ in our limit, i.e. what is

$$\lim_{3\to a} 2 \cdot 3 + 1 \,?$$

Of course $2\cdot 3 + 1 = 7$, but what does 7 do when 3 gets closer and closer to the number a? That's a silly question, because 3 is a constant and it doesn't "get closer" to some other number like a! If you ever see 3 get closer to another number then it's time to take a vacation.

On the other hand the variable a is free: you can assign it particular values, and its value will affect the value of the limit. For instance, if we set $a = 3$ (but leave x alone) then we get

$$\lim_{x\to 3} 2x + 1$$

and there's nothing strange about that (the limit is $2\cdot 3 + 1 = 7$, no problem.) You could substitute other values of a and you would get a different answer. In general you get $2a + 1$.