Abel means of the Fourier series of f:

$$\mathcal{A}_r f(\theta) = \hat{f}(0) + \sum_{n=1}^{\infty} r^n (\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta}) = \sum_{n\in\mathbb{Z}} r^{|n|}\hat{f}(n)e^{in\theta}.$$

- As Fourier coefficients are bounded then the series converges for every $0 < r < 1$

$$\begin{aligned}\mathcal{A}_r f(\theta) &= \sum_{n\in\mathbb{Z}} r^{|n|} \left(\frac{1}{2\pi} \int_0^{2\pi} f(\tau)e^{-in\tau}\, d\tau \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n\in\mathbb{Z}} r^{|r|} e^{in(\theta-\tau)} \right) f(\tau)\, d\tau.\end{aligned}$$

$$\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta} = 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} = \frac{1-r^n}{1-2r\cos\theta + r^2}.$$

Then

$$\begin{aligned}\mathcal{A}_r f(\theta) &= \int_0^{2\pi} P(re^{i\theta}, e^{i\tau})\, d\tau \\ &= Pf(re^{i\theta}).\end{aligned}$$

Corollary 1. *If f is continuous at θ, then $\mathcal{A}_r f(\theta) \to f(\theta)$ as $r \to 1$. If $f \in C(\mathbb{S})$, then $\mathcal{A}_r f$ converges uniformly to f on $\mathbb{S}$.*

Answering question 4

Let f be a Riemman integrable function on $\mathbb{S}$

1. If f is continuous at θ and its Fourier series converges at θ then it converges to $f(\theta)$.
   Proof: By Abel's theorem, if $s_n(\theta) \to L$ then $\mathcal{A}_r f(\theta) \to L$. But $\mathcal{A}_r f(\theta) \to f(\theta)$. □

2. If f is continuous at θ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(\theta) = 0$.
   Corollary: If f, g have the same Fourier coefficients and are both continuous at θ then $f(\theta) = g(\theta)$.

3. If f is continuous at θ and $\sum |\hat{f}(n)| \leq \infty$, then its Fourier series at θ does not converge to $f(\theta)$.
   By M-test: Uniformly to f if $f \in C(\mathbb{S})$. □