Theorem 2.7 ([5]). *Let $0 < p < \infty$. M is bounded on $\Lambda^p_u(w)$ if and only if there exists $q < p$ such that for some constant c and for every finite family of cubes and sets $(Q_j, E_j)_j$ with $E_j \subset Q_j$.*

$$\frac{W(u(\bigcup_j Q_j))}{W(u(\bigcup_j E_j))} \le c \max_j \left(\frac{|Q_j|}{|E_j|}\right)^q. \tag{2.4}$$

It is also mentioned in [5] that for a wide class of w, for instance for $w(t) = t^\alpha$, $\alpha > -1$, (2.4) is equivalent to the same condition but with a unique Q and $E \subset Q$. Thus, Theorem 2.7 represents a generalized version of Proposition 2.2.

We show that Theorems 2.3 and 2.4 and their proofs can be generalized with minor changes to the spaces $\Lambda^p_u(w)$. To be more precise, given weights u and w, we associate the function $\nu_{u,w}$ defined by

$$\nu_{u,w}(\lambda) = \inf_{\{Q_j\}} \inf_{\{E_j\}: E_j \subset Q_j,\ \min_j |E_j|/|Q_j| = \lambda} \frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))},$$

where the infimum is taken over all finite families of cubes $\{Q_j\}$ and over all families of sets $\{E_j\}$ such that $E_j \subset Q_j$ with $\min_j |E_j|/|Q_j| = \lambda$.

Theorem 2.8. *For any $p > 0$ we have*

$$\Phi_{\Lambda^p_u(w)}(\lambda) \asymp \frac{1}{\nu_{u,w}^{1/p}(\lambda)} \tag{2.5}$$

and

$$\alpha_{\Lambda^p_u(w)} = \frac{1}{p} \lim_{\lambda \to 0} \frac{\log(1/\nu_{u,w}(\lambda))}{\log(1/\lambda)}. \tag{2.6}$$

Theorem 2.9. *Let $0 < p < \infty$. Given weights u and w, the following statements are equivalent.*

(i) *M is bounded on $\Lambda^p_u(w)$;*

(ii) $\displaystyle\lim_{\lambda \to 0} \frac{\nu_{u,w}(\lambda)}{\lambda^p} = +\infty$;

(iii) $\displaystyle\lim_{\lambda \to 0} \frac{\log(1/\nu_{u,w}(\lambda))}{\log(1/\lambda)} < p$;

(iv) *if $\psi \in \mathcal{A}$, then for any finite family of cubes $\{Q_j\}$ and any family of sets $\{E_j\}$ with $E_j \subset Q_j$,*

$$\min_j \frac{|E_j|}{|Q_j|} \psi\left(\frac{|Q_j|}{|E_j|}\right) \le c \left(\frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))}\right)^{1/p}.$$

The first equivalence follows from the previous proposition, and the second one is trivial.

The following lemma shows that except for the trivial case $\Phi_X \equiv \infty$, Φ_X is equivalent to a finite non-increasing submultiplicative function near the origin. This is enough to give meaning to the limit in Definition 1.1 since, by Proposition 3.3, the limit defining α_X exists:

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)}.$$

Lemma 3.4. *Let X be any quasi-Banach function space. If $\Phi_X(\lambda_0) < \infty$, for some $\lambda_0 \in (0, 1/4^n]$ then there is a non-increasing, submultiplicative on $(0,1]$ function $\tilde{\Phi}_X$ such that $\tilde{\Phi}_X(1) = 1$, and*

$$c\Phi_X(\lambda) \le \tilde{\Phi}_X(\lambda) \le \Phi_X(\lambda) \quad (0 < \lambda < 1), \tag{3.11}$$

where c depends only on X.

Proof. It follows from (3.6) that

$$\|m_{2^n\lambda\xi} f\|_X \le \|m_\xi(m_\lambda f)\|_X \le \Phi_X(\xi)\|m_\lambda f\|_X \le \Phi_X(\xi)\Phi_X(\lambda)\|f\|_X,$$

and thus,

$$\Phi_X(2^n\lambda\xi) \le \Phi_X(\xi)\Phi_X(\lambda) \quad \left(\lambda < 1,\ \xi < \frac{1}{2^n}\right). \tag{3.12}$$

Set now

$$\tilde{\Phi}_X(\lambda) = \sup_{0<\xi<1} \frac{\Phi_X(\xi\lambda)}{\Phi_X(\xi)} \quad (0 < \lambda \le 1).$$

It is clear that $\tilde{\Phi}_X$ is submultiplicative on $(0,1]$ and $\tilde{\Phi}_X(1) = 1$. Next, $\tilde{\Phi}_X$ is non-increasing because Φ_X is so. Also, due to the fact that Φ_X is non-increasing, the left-hand inequality in (3.11) holds trivially with $c_1 = 1/\Phi_X(1-)$. Further, it follows from (3.12) that

$$\frac{\Phi_X(\xi\lambda)}{\Phi_X(\xi)} \le \frac{\Phi_X(\xi/2^n)}{\Phi_X(\xi)}\Phi(\lambda) \le \Phi_X(1/4^n)\Phi(\lambda),$$

which proves the right-hand inequality in (3.11) with $c_2 = \Phi_X(1/4^n)$. Observe that c_2 is finite since $\Phi_X(\lambda_0) < \infty$, $0 < \lambda_0 \le 1/4^n$. ◻

4. Proof of the Main Results

Denote $M^2 f = MMf$. We start with the following simple lemma.

Therefore $\lim_{\lambda \to 0} \lambda \Phi_X(\lambda) = 0$, which proves (i) $\Rightarrow$ (iv).

Assume now that (ii) holds. This means that there are constants $c > 0$ and $\delta < 1$ such that for any f,

$$\|m_\lambda f\|_X \le c\lambda^{-\delta}\|f\|_X. \tag{4.2}$$

We next observe that for any cube Q,

$$\frac{1}{|Q|}\int_Q |f| = \int_0^1 (f\chi_Q)^*(\lambda|Q|)\,\mathrm{d}\lambda,$$

and hence,

$$Mf(x) \le \int_0^1 m_\lambda f(x)\,\mathrm{d}\lambda \le \sum_{i=1}^{\infty} 2^{-i} m_{2^{-i}} f(x).$$

From this and from (3.8) along with (4.2), we obtain

$$\|Mf\|_X \le \Big\|\sum_{i=1}^{\infty} 2^{-i} m_{2^{-i}} f\Big\|_X \le 4^{1/\rho}\Big(\sum_{i=1}^{\infty} \|2^{-i} m_{2^{-i}} f\|_X^{\rho}\Big)^{1/\rho}$$

$$\le c\Big(\sum_{i=1}^{\infty} 2^{-(1-\delta)\rho i}\Big)^{1/\rho}\|f\|_X \le c'\|f\|_X.$$

This completes the proof of (ii) $\Rightarrow$ (i).

Let us show now that if the space X is r-i space, then $\alpha_X = \bar{\alpha}_X$. Consider the spherically symmetric rearrangement of f defined by

$$f^\star(x) = f^*(v_n|x|^n),$$

where v_n is the volume of the unit ball. Note that the functions f and $f^\star$ are equimeasurable. It follows from (3.5) that

$$(D_{(2^n\lambda)^{1/n}} f)^\star(x) \le (m_\lambda f)^\star(x) \le (D_{(\lambda/3^n)^{1/n}} f)^\star(x).$$

Therefore,

$$\|D_{(2^n\lambda)^{1/n}} f\|_X \le \|m_\lambda f\|_X \le \|D_{(\lambda/3^n)^{1/n}} f\|_X$$

and

$$h_X\Big(\frac{1}{2^n\lambda}\Big) \le \Phi_X(\lambda) \le h_X\Big(\frac{3^n}{\lambda}\Big).$$

From this and from the definitions (1.2) and (3.9), we readily obtain that $\alpha_X = \bar{\alpha}_X$. ◻

[8] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238. MR 1976842 (2004c:42039)

[9] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), 245–253. MR 2057643 (2005k:42048)

[10] ———, *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (2005), 657–700, http://dx.doi.org/10.1016/j.bulsci.2003.10.003. MR 2166733 (2006e:46032) (English, with English and French summaries)

[11] J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, ISBN 0-444-87804-1, Notas de Matemática [Mathematical Notes], 104. MR 807149 (87d:42023)

[12] Loukas Grafakos and Nigel Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180, http://dx.doi.org/10.1007/PL00004426. MR 1812822 (2002a:46032)

[13] Tadeusz Iwaniec and Gaven Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2001, ISBN 0-19-850929-4. MR 1859913 (2003c:30001)

[14] N.J. Kalton, *Endomorphisms of symmetric function spaces*, Indiana Univ. Math. J. **34** (1985), 225–247, http://dx.doi.org/10.1512/iumj.1985.34.34014. MR 783913 (86a:47052)

[15] N.J. Kalton, N.T. Peck, and J.W. Roberts, *An F-space Sampler*, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984, ISBN 0-521-27585-7. MR 808777 (87c:46002)

[16] R.A. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71** (1981/82), 277–284. MR 667316 (83k:42019)

[17] V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co. Inc., River Edge, NJ, 1991, ISBN 981-02-0612-7. MR 1156767 (93g:42013)

[18] A.K. Lerner, *On some pointwise inequalities*, J. Math. Anal. Appl. **289** (2004), 248–259, http://dx.doi.org/10.1016/j.jmaa.2003.09.051. MR 2020540 (2004k:42023)

[19] G.G. Lorentz, *Majorants in spaces of integrable functions*, Amer. J. Math. **77** (1955), 484–492, http://dx.doi.org/10.2307/2372636. MR 0070981 (17,65a)

[20] Stephen J. Montgomery-Smith, *The Hardy operator and Boyd indices*, Proc. Probability, Columbia, MO, 1994, Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, 1996, pp. 359–364. MR 1358172 (97a:46037)

[21] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226, http://dx.doi.org/10.2307/1995882. MR 0293384 (45 #2461)

[22] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983, ISBN 3-540-12706-2. MR 724434 (85m:46028)

[23] A. Nekvinda, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$*, Math. Inequal. Appl. **7** (2004), 255–265. MR 2057644 (2005f:42045)

[24] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 471–473, XL. MR 0088682 (19,562d) (English, with Russian summary)

[25] T. Shimogaki, *Hardy-Littlewood majorants in function spaces*, J. Math. Soc. Japan **17** (1965), 365–373. MR 0182875 (32 #357)