Hence with $\Omega = \cup_{Q\in\mathcal{Q}} 10Q$ we have

$$\|\sum_Q Tb_Q\|_{L^1(\mathbb{R}^d\setminus\Omega)} \lesssim \|\omega\|_{\mathrm{Dini}} \sum_{Q\in\mathcal{Q}} |Q| \lesssim \|\omega\|_{\mathrm{Dini}} \|f\|_1$$

Therefore

$$\begin{aligned} |\{Tf > 1\}| &\le |\Omega| + |\{\sum_Q Tb_Q > 1/2\} \cap \Omega^c| + |\{Tg > 1/2\}| \\ &\lesssim \|f\|_1 + \|\sum_Q Tb_Q\|_{L^1(\mathbb{R}^d\setminus\Omega)} + \|Tg\|_2^2 \\ &\lesssim (1 + \|\omega\|_{\mathrm{Dini}})\|f\|_1, \end{aligned}$$

where we have used $\|g\|_2^2 \le \|g\|_1\|g\|_\infty \lesssim \|f\|_1$. □

5.2 Cotlar's inequality

Define the maximally truncated operator

$$T_\sharp f(x) := \sup_{\varepsilon>0, |x-x'|\le\varepsilon/2} \int_{B(x',\varepsilon)^c} K(x',y)f(y)\mathrm{d}y.$$

This maximal truncation is usually considered without the supremum in x' (i.e. with $x' = x$), but the above version is more convenient for us.

Lemma 5.5 (Cotlar's inequality)**.**

$$T_\sharp f \lesssim_{d,\delta} (\|T\|_{L^2\to L^2} + \|\omega\|_{\mathrm{Dini}})Mf + M_\delta Tf. \tag{5.6}$$

Here $M_\delta f = (M(f^\delta))^{1/\delta}$, $0 < \delta < 1$, where M is the usual Hardy–Littlewood maximal function.

In particular $T_\sharp$ has weak type $(1,1)$.

Proof. For $x', x'' \in B(x, \varepsilon/2)$ write

$$\begin{aligned} \int_{B(x',\varepsilon)^c} K(x',y)f(y)\mathrm{d}y = \int_{B(x',\varepsilon)^c\setminus B(x,2\varepsilon)^c} K(x',y)f(y)\mathrm{d}y + \int K(x'',y)(f\mathbf{1}_{B(x,2\varepsilon)^c})(y)\mathrm{d}y \\ - \int (K(x'',y) - K(x',y))\mathbf{1}_{B(x,2\varepsilon)^c} f(y)\mathrm{d}y. \end{aligned}$$

The first term is estimated using the kernel bound by $C_K Mf(x)$. The last term is estimated by

$$\sum_{k>0} \int_{2^k\varepsilon\le|x-y|<2^{k+1}\varepsilon} |K(x'',y) - K(x',y)||f(y)|\mathrm{d}y \lesssim_d \sum_{k>0} \int_{2^k\varepsilon\le|x-y|<2^{k+1}\varepsilon} \frac{\omega(2^{-k})}{(2^k\varepsilon)^d} |f(y)|\mathrm{d}y \lesssim \|\omega\|_{\mathrm{Dini}} Mf(x).$$

The middle term equals

$$T(f\mathbf{1}_{B(x,2\varepsilon)^c})(x'') = T(f)(x'') - T(f\mathbf{1}_{B(x,2\varepsilon)})(x''),$$

where we have used that T is associated to K and linearity of T. In both terms we take the L^δ average over $x'' \in B := B(x, \varepsilon/2)$. The contribution of the former term is then clearly bounded by $M_\delta Tf(x)$. The contribution of the latter term is bounded by

$$(\fint_B |T(\mathbf{1}_{4B}f)|^\delta)^{1/\delta} \lesssim |B|^{-1} \|T(\mathbf{1}_{4B}f)\|_{L^{1,\infty}(B)} \le \|T\|_{L^1\to L^{1,\infty}} |B|^{-1} \|\mathbf{1}_{4B}f\|_{L^1} \lesssim \|T\|_{L^1\to L^{1,\infty}} Mf(x),$$

and we conclude using Lemma 5.4. □

Exercise 5.7. Replace $M_\delta Tf$ in (5.6) by $\mathcal{M}_{1/2}Tf$, where

$$\mathcal{M}_\lambda f(x) = \sup_{x\in Q} (f\mathbf{1}_Q)^*(\lambda|Q|)$$

and f^* denotes the non-increasing rearrangement of f.

5.3 Marcinkiewicz interpolation theorem, $L^{p,\infty}$ version

We need the following version of the Marcinkiewicz interpolation theorem in which the conclusion is a bound on a weak L^p space.

Theorem 5.8. *Let T be a quasisubadditive operator and assume $T : L^{p_j} \to L^{p_j,\infty}$ for $j = 0, 1$ with $1 \le p_0 < p_1 \le \infty$. Let $0 < \theta < 1$ and $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. Then $T : L^{p_\theta,\infty} \to L^{p_\theta,\infty}$.*

Proof. Similarly as in the proof of the strong type estimate split $f = f_{0,\lambda} + f_{1,\lambda}$ with $f_{1,\lambda} = f 1_{|f| \le \lambda}$. Then

$$\begin{aligned}
\{|Tf| > \eta\} &\le \{|Tf_{0,\lambda}| > \eta/(2C)\} + \{|Tf_{1,\lambda}| > \eta/(2C)\} \\
&\lesssim \eta^{-p_0} \|Tf_{0,\lambda}\|_{p_0,\infty}^{p_0} + \eta^{-p_1} \|Tf_{1,\lambda}\|_{p_1,\infty}^{p_1} \\
&\lesssim \eta^{-p_0} \|f_{0,\lambda}\|_{p_0}^{p_0} + \eta^{-p_1} \|f_{1,\lambda}\|_{p_1}^{p_1} \\
&\le \eta^{-p_0} \int_{|f|>\lambda} |f|^{p_0} + \eta^{-p_1} \int_{|f|\le\lambda} |f|^{p_1} \\
&\le \eta^{-p_0} \sum_{k\ge 0} \int_{|f|\sim 2^k\lambda} |f|^{p_0} + \eta^{-p_1} \sum_{k\le 0} \int_{|f|\sim 2^k\lambda} |f|^{p_1} \\
&\le \eta^{-p_0} \sum_{k\ge 0} (2^k\lambda)^{p_0-p_\theta} \|f\|_{p_\theta,\infty} + \eta^{-p_1} \sum_{k\le 0} (2^k\lambda)^{p_1-p_\theta} \|f\|_{p_\theta,\infty}.
\end{aligned}$$

Since $p_0 < p_\theta < p_1$, both series are geometric and dominated by the $k = 0$ terms. Hence

$$\{|Tf| > \eta\} \lesssim \eta^{-p_0} (\lambda)^{p_0-p_\theta} \|f\|_{p_\theta,\infty} + \eta^{-p_1} (\lambda)^{p_1-p_\theta} \|f\|_{p_\theta,\infty}.$$

Choosing $\lambda = \eta$ we obtain the claim

$$\{|Tf| > \eta\} \lesssim \eta^{-p_\theta} \|f\|_{p_\theta,\infty}.$$

□

Corollary 5.9. *The maximal operator M_δ is bounded on $L^{1,\infty}$ for $0 < \delta < 1$.*

Proof. By Theorem 5.8 the Hardy–Littlewood maximal operator M is bounded on $L^{1/\delta,\infty}$. Hence

$$\|M_\delta f\|_{1,\infty} = \|M(f^\delta)\|_{1/\delta,\infty} \lesssim \|f^\delta\|_{1/\delta,\infty} = \|f\|_{1,\infty}.$$

□

6 Sparse domination of CZ operators

The fierst proof of sharp weighted estimates for CZ operators was quite complicated [Hyt12]. Many simplifications have been made since then The two key simplifications were the introduction of sparse domination by Lerner [Ler13] and a simple algorithm for constructing sparse collections by Lacey [Lac15], a streamlined version of which appears in [HRT15]. We have followed [Ler16].

The main example that I am aware of where sharp weighted estimates are useful is the regularity theory for solutions of the Beltrami equation in [AIS01].

7 A_∞ weights

Let $\mathscr{D}$ be a dyadic grid (in $\mathbb{R}^d$) and M the associated dyadic maximal operator. The associated A_∞ characteristic is defined by

$$[w]_{A_\infty} := \sup_{Q\in\mathscr{D}} w(Q)^{-1} \int_Q M(w\mathbf{1}_Q).$$

Consider now the case $v = w$ that corresponds to (7.6). By Lemma 7.3 we obtain

$$w\{Mw > \lambda\} \le 2^d \lambda |\{Mw > \lambda\}|.$$

Hence using (7.7) and (7.5) we obtain

$$\begin{aligned}\varepsilon \int_{(w)_{Q_0}}^{\infty} \lambda^{\varepsilon-1} w\{Mw > \lambda\} \mathrm{d}\lambda \le 2^d \varepsilon \int_{(w)_{Q_0}}^{\infty} \lambda^{\varepsilon} |\{Mw > \lambda\}| \mathrm{d}\lambda &\le \frac{2^d \varepsilon}{1+\varepsilon} \int_{Q_0} (Mw)^{1+\varepsilon} \\ &\le \frac{2^{d+1}[w]_{A_\infty} \varepsilon}{1+\varepsilon} |Q_0| (w)_{Q_0}^{1+\varepsilon} = |Q_0|(w)_{Q_0}^{1+\varepsilon}\end{aligned}$$

by the choice of ε. The conclusion follows. □

Corollary 7.8 (Open property). *Let $1 < p < \infty$ and $w \in A_p$. Then $[w]_{A_{\tilde p}} \lesssim [w]_{A_p}$, where $\tilde p = p - \frac{p-1}{2^{d+1}[v]_{A_\infty}} < p$ and v is the dual weight: $w^{1/p} v^{1/p'} \equiv 1$.*

Proof. The exponent $\tilde p$ is chosen in such a way that $1 + \varepsilon = (p/p')(\tilde p'/\tilde p)$, where ε is as in Lemma 7.4 for the weight v. Consider the dual weight $\tilde v = w^{-\tilde p'/\tilde p}$. Then by (7.6) applied to the weight v we have

$$(\tilde v)_Q = (v^{1+\varepsilon})_Q \le 2(v)_Q^{1+\varepsilon}.$$

Hence for every $Q \in \mathscr{D}$ we have

$$(w)_Q (\tilde v)_Q^{\tilde p/\tilde p'} \lesssim (w)_Q (v)_Q^{(1+\varepsilon)\tilde p/\tilde p'} = (w)_Q (v)_Q^{p/p'} \le [w]_{A_p}.$$ □

7.1 Embedding of A_∞ into A_p

We call a weight *(C_{db}-)doubling* if

$$w(2Q) \le C_{db} w(Q)$$

for some *doubling constant* $C_{db} < \infty$ and all cubes $Q \subset \mathbb{R}^d$.

It is not hard to show that A_p weights are doubling if $p < \infty$. The case $p = \infty$ is more subtle.

Exercise 7.9. Find a weight that is A_∞ with respect to the standard dyadic filtration but not $A_\infty(\mathbb{R}^d)$.

Exercise 7.10. Find a weight on $\mathbb{R}$ that is A_∞ with respect to the three 1/3-shifted dyadic grids but not $A_\infty(\mathbb{R})$.

To combat these difficulties we define the $A_\infty(\mathbb{R}^d)$ by

$$[w]_{A_\infty(\mathbb{R}^d)} = \sup_Q w(Q)^{-1} \int_Q M(w \mathbf{1}_Q),$$

where the supremum is taken over *all* non-empty axis-parallel cubes in $\mathbb{R}^d$.

Lemma 7.11. *For every $d \ge 1$ there exists $C = C(d)$ such that for every $w \in A_\infty(\mathbb{R}^d)$ we have*

$$C_{db}(w) \le C^{C^{[w]_{A_\infty(\mathbb{R}^d)}}}.$$

The converse is not true: there exist doubling weights that are not A_∞, see [FM74] (a different version of the A_∞ condition was used there).

Consider first the case $p < 2$, so that $\lfloor p \rfloor = 1$ and $1/p > 1/p'$. Then we estimate this by

$$= \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \Big(w(Q_1)^{-1} \sum_{Q \subseteq Q_1} (v)_Q (w)_Q |Q| \Big)^{\{p\}}$$

$$\leq [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \Big(w(Q_1)^{-1} \sum_{Q \subseteq Q_1} (v)_Q^{1-p/p'} |Q| \Big)^{\{p\}}$$

Using Lemma 8.4

$$\lesssim [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} (v)_{Q_1} w(Q_1) \Big(w(Q_1)^{-1} (v)_{Q_1}^{1-p/p'} |Q_1| \Big)^{\{p\}}$$

$$= [v,w]^{\{p\}(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}^{1+(1-p/p')\{p\}} (w)_{Q_1}^{1-\{p\}}$$

$$\leq [v,w]^{(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}^{1+(1-p/p')\{p\}-(p/p')(1-\{p\})}$$

$$= [v,w]^{(p/p',1)} \sum_{F \supseteq Q_1} |Q_1| (v)_{Q_1}$$

and using sparseness of $\mathscr{Q}$

$$\lesssim [v,w]^{(p/p',1)} [v]_{A_\infty} v(F).$$

Consider now the case $p \geq 2$, so that $1/p \leq 1/p'$. Then we estimate

$$= \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \Big(w(Q_{\lfloor p \rfloor})^{-1} \sum_{Q \subseteq Q_{\lfloor p \rfloor}} |Q| (v)_Q (w)_Q \Big)^{\{p\}}$$

$$\leq [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \Big(w(Q_{\lfloor p \rfloor})^{-1} \sum_{Q \subseteq Q_{\lfloor p \rfloor}} |Q| (w)_Q^{1-p'/p} \Big)^{\{p\}}$$

Using Lemma 8.4

$$\lesssim [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) \Big(w(Q_{\lfloor p \rfloor})^{-1} |Q_{\lfloor p \rfloor}| (w)_{Q_{\lfloor p \rfloor}}^{1-p'/p} \Big)^{\{p\}}$$

$$= [v,w]^{\{p\}(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor}} w(Q_{\lfloor p \rfloor}) (w)_{Q_{\lfloor p \rfloor}}^{-\{p\}p'/p}$$

$$\leq [v,w]^{\{p\}(1,p'/p)+(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - 1}} |Q_{\lfloor p \rfloor}| (w)_{Q_{\lfloor p \rfloor}}^{1-\{p\}p'/p-p'/p}$$

Using Lemma 8.4 again

$$\lesssim [v,w]^{\{p\}(1,p'/p)+(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor - 1}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - 1}} |Q_{\lfloor p \rfloor - 1}| (w)_{Q_{\lfloor p \rfloor - 1}}^{1-\{p\}p'/p-p'/p}.$$

Continuing in this manner we obtain inductively

$$\lesssim [v,w]^{\{p\}(1,p'/p)+m(1,p'/p)} \sum_{F \supseteq Q_1 \supseteq \cdots \supseteq Q_{\lfloor p \rfloor - m}} (v)_{Q_1} \cdots (v)_{Q_{\lfloor p \rfloor - m}} |Q_{\lfloor p \rfloor - m}| (w)_{Q_{\lfloor p \rfloor - m}}^{1-\{p\}p'/p-mp'/p}.$$

For $m = \lfloor p \rfloor - 1$ this gives the estimate

$$\lesssim [v,w]^{\{p\}(1,p'/p)+(\lfloor p \rfloor - 1)(1,p'/p)} \sum_{F \supseteq Q_1} (v)_{Q_1} |Q_1| (w)_{Q_{\lfloor p \rfloor - m}}^{1-\{p\}p'/p-(\lfloor p \rfloor - 1)p'/p}$$