Proof. We say that φ is radial if there is a function $u\colon [0, +\infty) \to \mathbf{R}$ such that $\varphi(x) = u(|x|)$ for every $x \in \mathbf{R}^d$. Also we say that a radial function φ is decreasing if u is decreasing.

The function u is measurable, hence there is an increasing sequence of simple functions (u_n) such that $u_n(t)$ converges to $u(t)$ for every $t \geq 0$. In this case, since u is decreasing, it is possible to choose each u_n

$$u_n(t) = \sum_{j=1}^{N} h_j \, \chi_{[0,t_j]}(t),$$

where $0 < t_1 < t_2 < \cdots < t_N$ and $h_j > 0$ and the natural number N depends on n.

Now the proof is straightforward. Let $\varphi_n(x) = u_n(|x|)$. By the monotone convergence theorem

$$|\varphi * f(x)| \leq \varphi * |f|(x) = \lim_n \varphi_n * |f|(x).$$

Therefore

$$\varphi_n * |f|(x) = \sum_{j=1}^{N} h_j \int_{B(x,t_j)} |f(y)| \, dy.$$

We can replace the ball $B(x, t_j)$ by the cube with center x and side $2t_j$. The quotient between the volume of the ball and the cube is bounded by a constant. Thus

$$\varphi_n * |f|(x) \leq \sum_{j=1}^{N} h_j \mathfrak{m}\big(Q(x_j, t_j)\big) \cdot \mathcal{M}f(x) \leq C_d \|\varphi\|_1 \mathcal{M}f(x).$$

□