Proof. We say that φ is radial if there is a function $u:[0,+\infty)\to \mathbf{R}$ such that $\varphi(x)=u(|x|)$ for every $x\in\mathbf{R}^d$. Also we say that a radial function φ is decreasing if u is decreasing.

The function u is measurable, hence there is an increasing sequence of simple functions (u_n) such that $u_n(t)$ converges to u(t) for every $t \geq 0$. In this case, since u is decreasing, it is possible to choose each u_n

$$u_n(t) = \sum_{j=1}^{N} h_j \chi_{[0,t_j]}(t),$$

where $0 < t_1 < t_2 < \cdots < t_N$ and $h_j > 0$ and the natural number N depends on n.

Now the proof is straightforward. Let $\varphi_n(x) = u_n(|x|)$. By the monotone convergence theorem

$$|\varphi * f(x)| \le \varphi * |f|(x) = \lim_{n} \varphi_n * |f|(x).$$

Therefore

$$\varphi_n * |f|(x) = \sum_{j=1}^{N} h_j \int_{B(x,t_j)} |f(y)| dy.$$

We can replace the ball $B(x,t_j)$ by the cube with center x and side $2t_j$. The quotient between the volume of the ball and the cube is bounded by a constant. Thus

$$\varphi_n * |f|(x) \leq \sum_{j=1}^N h_j \mathfrak{m} \big(Q(x_j, t_j) \big) \cdot \mathcal{M} f(x) \leq C_d \|\varphi\|_1 \mathcal{M} f(x).$$
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